

# Tangent categories of algebras over operads

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## ABSTRACT

The abstract cotangent complex formalism, as developed in the  $\infty$ -categorical setting by Lurie, brings together classical notions such as parametrized spectra, obstruction theory and deformation theory in a unified setting. When the  $\infty$ -category at hand consists of algebras over a nice  $\infty$ -operad in a stable  $\infty$ -category, the target category of the abstract cotangent complex can be identified with the associated  $\infty$ -category of operadic modules, by work of Basterra–Mandell, Schwede and Lurie. In this paper we develop the model categorical counterpart of this identification and extend it to the case of algebras over an enriched operad which is not necessarily simplicial, taking values in a model category which is not necessarily stable. Such a comparison result can be used, in particular, to identify the cotangent complex and Quillen cohomology of enriched categories, an application we take up in a subsequent paper.

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## 1. Introduction

A ubiquitous theme in mathematics is the contrast between linear and non-linear structures. In algebraic settings, linear objects such as vector spaces, abelian groups, and modules tend to

have a highly structured and accessible theory, while non-linear objects, such as groups, rings, or algebraic varieties are more wild, and harder to analyze. For example, understanding maps  $X \rightarrow Y$  between algebraic varieties is a complicated non-linear problem, but to understand the sections of a vector bundle on  $X$  is much more accessible. Non-linear objects often admit interesting **linear invariants** which are fairly computable and easy to manipulate. Homological algebra then typically enters the picture, extending a given invariant to a collection of derived ones.

An extremely useful idea which can be applied in a variety of contexts is to try to translate a non-linear problem into a collection of **linear problems**. For example, maps  $f : X \rightarrow Y$  of algebraic varieties which are very close to a fixed map  $f_0 : X \rightarrow Y$  can be understood by studying linear problems involving the pulled back tangent bundle  $f_0^*TY$ . In a different direction, many problems involving solvable groups can be reduced to an iteration of linear problems involving abelian groups.

To streamline this idea one would like to have a formal framework to understand what linear objects are and how one can “linearize” a given non-linear object. One way to do so is the following. Let  $\mathbf{Ab}$  denote the category of abelian groups. A locally presentable category  $\mathcal{C}$  is called **additive** if it is tensored over  $\mathbf{Ab}$ . We note that in this case the tensoring is essentially unique and induces a natural enrichment of  $\mathcal{C}$  in  $\mathbf{Ab}$ . If  $\mathcal{D}$  is a locally presentable category then there exists a universal additive category  $\mathbf{Ab}(\mathcal{D})$  receiving a colimit preserving functor  $\mathbb{Z} : \mathcal{D} \rightarrow \mathbf{Ab}(\mathcal{D})$ . The category  $\mathbf{Ab}(\mathcal{D})$  can be described explicitly as the category of **abelian group objects** in  $\mathcal{D}$ , namely, objects  $M \in \mathcal{D}$  equipped with maps  $u : *_{\mathcal{D}} \rightarrow M$ ,  $m : M \times M \rightarrow M$  and  $\text{inv} : M \rightarrow M$  satisfying (diagrammatically) all the axioms of an abelian group. We may then identify  $\mathbb{Z} : \mathcal{D} \rightarrow \mathbf{Ab}(\mathcal{D})$  with the functor which sends  $A$  to the free abelian group  $\mathbb{Z}A$  generated from  $A$ , or the **abelianization** of  $A$ .

Although this yields a procedure for replacing objects in a category by linear objects, such linear approximations tend to be too coarse if one is studying maps between objects, rather than the objects themselves. Instead, when studying maps  $f : B \rightarrow A$  one is often interested in linear invariants of  $B$  **over**  $A$ . A way to do this formally was developed by Beck in [Bec67], where he defined the notion of a **Beck module** over an object  $A$  (say, in a locally presentable category  $\mathcal{D}$ ) to be an abelian group object of the slice category  $\mathcal{D}_{/A}$ . Simple as it is, this definition turns out to capture many well-known instances of “linear objects over a fixed object  $A$ ”. For example, if  $G$  is a group and  $M$  is a  $G$ -module then the semi-direct product  $M \rtimes G$  carries a natural structure of an abelian group object in  $\mathbf{Grp}_{/G}$ . One can then show that the association  $M \mapsto M \rtimes G$  determines an equivalence between the category of  $G$ -modules and the category of abelian group objects in  $\mathbf{Grp}_{/G}$ . If  $\mathcal{D} = \mathbf{Ring}$  is the category of associative unital rings then one may replace the formation of semi-direct products with that of **square-zero extensions**, yielding an equivalence between the notion of a Beck module over a ring  $R$  and the notion of an  $R$ -bimodule, i.e., an abelian group equipped with compatible left and right actions of  $R$  (see [Qui70]). When  $R$  is a commutative ring the corresponding notion of a Beck module reduces to the usual notion of an  $R$ -module. For an example of a different nature, if  $\mathcal{D}$  is a Grothendieck topos and  $X \in \mathcal{D}$  is an object then  $\mathcal{D}_{/X}$  is also a topos and there exists a small site  $\mathcal{T}_X \subseteq \mathcal{D}_{/X}$  such that  $\mathcal{D}_{/X}$  is equivalent to the category of sheaves of sets on  $\mathcal{T}_X$ . A Beck module over  $X$  then turns out to be the same as a sheaf of abelian groups on  $\mathcal{T}_X$ . We can therefore summarize by saying that the notion of a Beck module provides one with a robust abstract framework which indicates, in a given non-linear context, what are the relevant linearized counterparts.

In the realm of algebraic topology, one linearizes spaces by evaluating **cohomology theories**

on them. This approach is closely related to the approach of Beck: indeed, by the classical Dold-Thom theorem one may identify the ordinary homology groups of a space  $X$  with the homotopy groups of the free abelian group generated from  $X$  (considered, for example, as a simplicial abelian group). The quest for more refined invariants has led to the consideration of **generalized cohomology theories** and their classification via homotopy types of **spectra**. The extension of cohomological invariants from ordinary cohomology to generalized cohomology therefore highlights spectra as a natural extension of the notion of “linearity” provided by abelian groups.

The passage from abelian groups to spectra is a substantial one, even from the homotopy-theoretic point of view. Indeed, one should observe that in a homotopical context there is a natural notion of an  $\mathbb{E}_\infty$ -group object, obtained by interpreting the axioms of an abelian group not strictly, but up to coherent homotopy. For spaces, it turns out that specifying an  $\mathbb{E}_\infty$ -group structure on a given space  $X_0$  is equivalent to specifying, for every  $n \geq 1$ , an  $(n-1)$ -connected space  $X_n$ , together with a weak equivalence  $X_{n-1} \xrightarrow{\simeq} \Omega X_n$ . Such a datum is also known as a **connective spectrum**, and naturally extends to the general notion of a spectrum by removing the connectivity conditions on  $X_n$ . This passage from connective spectra (or  $\mathbb{E}_\infty$ -group objects) to spectra should be thought of as an extra linearization step that is possible in a homotopical setting, turning additivity into **stability**. It has the favorable consequence that kernels and cokernels of maps become equivalent up to a shift. Using stability as the fundamental form of linearity is also the starting point for the theory of **Goodwillie calculus**, which extends the notion of stability to give meaningful analogues to higher order approximations, derivatives and Taylor series for functors between  $\infty$ -categories. Replacing the category of abelian groups with the  $\infty$ -category of spectra means we should replace the notion of an additive category with the notion of a **stable  $\infty$ -category**. The operation associating to a locally presentable category  $\mathcal{D}$  the additive category  $\text{Ab}(\mathcal{D})$  of abelian group objects in  $\mathcal{D}$  is now replaced by the operation which associates to a presentable  $\infty$ -category  $\mathcal{D}$  its  $\infty$ -category  $\text{Sp}(\mathcal{D})$  of **spectrum objects in  $\mathcal{D}$** , which is the universal stable presentable  $\infty$ -category receiving a colimit preserving functor  $\Sigma_+^\infty : \mathcal{D} \rightarrow \text{Sp}(\mathcal{D})$ .

The construction of Beck modules as a form of linearization and the homotopical notion of linearization through spectra were brought together in [Lur14, §7.3] under the the framework of the **abstract cotangent complex formalism**. Given a presentable  $\infty$ -category  $\mathcal{D}$  and an object  $A \in \mathcal{D}$ , one may define the analogue of a Beck module to be a **spectrum object** in the slice  $\infty$ -category  $\mathcal{D}_{/A}$ . With a geometric analogy in mind, if we consider objects  $B \rightarrow A$  of  $\mathcal{D}_{/A}$  as paths in  $\mathcal{D}$ , then we may consider spectrum objects in  $\mathcal{D}_{/A}$  as “infinitesimal paths”, or “tangent vectors” at  $A$ . As in [Lur14], we will consequently refer to  $\text{Sp}(\mathcal{D}_{/A})$  as the **tangent  $\infty$ -category** at  $A$ , and denote it by  $\mathcal{T}_A \mathcal{D}$ . Just like the tangent space is a linear object, we may consider  $\mathcal{T}_A \mathcal{D}$  as linear, being a stable  $\infty$ -category. This analogy is helpful in many of the contexts in which linearization plays a significant role. Furthermore, it is often useful to assemble the various tangent categories into a global object, which is then known as the **tangent bundle  $\infty$ -category**  $\mathcal{T}\mathcal{D}$ .

An important consequence of the linearization process encompassed in the cotangent complex formalism is that it allows one to produce cohomological invariants of a given object  $A \in \mathcal{D}$  in a universal way. The resulting cohomology groups are known as **Quillen cohomology groups**, and take their coefficients in the tangent  $\infty$ -category  $\mathcal{T}_A \mathcal{D}$ . In order to study Quillen cohomology effectively one should therefore understand the various tangent  $\infty$ -categories  $\mathcal{T}_A \mathcal{D}$  in reasonably concrete terms.

One of the main theorems of [Lur14, §7.3] identifies the tangent categories of algebras in a presentable stable  $\infty$ -category over a given (unital, coherent)  $\infty$ -operad with the corresponding operadic module categories. Earlier results along these lines were obtained in [Sch97] and [BM05]. We note that this is indeed analogous to what happens in the discrete case of Beck modules. For example, if  $\mathcal{C}$  is the  $\infty$ -category of  $\mathbb{E}_\infty$ -ring spectra then the above results identify the tangent  $\infty$ -category  $\mathcal{T}_A \mathcal{C}$  at a given  $\mathbb{E}_\infty$ -ring spectrum  $A$  with the  $\infty$ -category of  $A$ -modules in spectra. This allows one, for example, to identify the Quillen cohomology theory of  $\mathbb{E}_\infty$ -ring spectra with topological André-Quillen cohomology.

Our main motivation in this paper is to generalize these results to the setting where the algebras take values in an  $\infty$ -category which is **not necessarily stable**. This allows one, in principle, to compute tangent categories in much more general algebraic settings, and consequently identify the category of coefficients for the associated Quillen cohomology. For example, it allows one to compute tangent categories and Quillen cohomology of objects such as **simplicial categories**, or more generally enriched categories, an application which is described in a subsequent paper [HNP16].

For various reasons we found it convenient to work in the setting of combinatorial model categories. Using this setting our main result can be formulated as follows (see Corollary 4.3.1 below).

**THEOREM 1.0.1.** *Let  $\mathcal{M}$  be a differentiable, left proper, combinatorial symmetric monoidal model category and let  $\mathcal{P}$  be an admissible colored symmetric operad in  $\mathcal{M}$ . Let  $A$  be a fibrant  $\mathcal{P}$ -algebra such that the enveloping operad  $\mathcal{P}^A$  is stably admissible and  $\Sigma$ -cofibrant. Then the Quillen adjunction*

$$\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}}(\mathcal{M}) \xrightleftharpoons[\tau]{\tau} \mathcal{T}_A \mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M}) \quad (1.0.1)$$

*induced by the free-forgetful adjunction is a Quillen equivalence.*

When every object in  $\mathcal{M}$  is cofibrant and  $\mathcal{P}$  is a cofibrant single-colored operad then the enveloping operad  $\mathcal{P}^A$  is stably admissible and  $\Sigma$ -cofibrant for every  $\mathcal{P}$ -algebra  $A$  by work of Fresse ([Fre09], see Remark 4.2.2). This is also true when  $\mathcal{M}$  is the category of simplicial sets and  $\mathcal{P}$  is an arbitrary cofibrant colored open by work of Rezk ([Rez02]).

The construction of stabilizations in the model categorical setting can be done in many ways (see [Hov01],[Lyd98]). However, most constructions assume given a self Quillen adjunction realizing the loop-suspension adjunction. Unfortunately, such a structure is not always readily available. With an eye toward future applications we hence chose to dedicate the first section of this paper to setting up a more flexible model for the stabilization (based on ideas of Heller and Lurie) in which one does not need to assume such additional structures.

Theorem 1.0.1 identifies, under suitable assumptions, the tangent model category at a given operadic algebra  $A$  with the tangent category to  $A$  in the model category of  $A$ -modules. This latter tangent category can be further simplified into something which resembles a functor category with stable codomain. To make this idea precise it is useful to exploit the global point of view obtained by assembling the various tangent categories into a **tangent bundle**. This can be done in the model categorical setting by using the machinery of [HP15], and is carried out in §3. The final identification of  $\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}}(\mathcal{M})$  then takes the following form (see Corollary 4.3.4 below):

**COROLLARY 1.0.2.** *Let  $\mathcal{M}, \mathcal{P}$  and  $A$  be as in Theorem 1.0.1. Then we have a natural Quillen*

equivalence

$$\mathcal{T}_A \text{Alg}_{\mathcal{P}}(\mathcal{M}) \xrightleftharpoons[\tau]{\simeq} \text{Fun}_{/\mathcal{M}}^{\mathcal{M}}(\mathcal{P}_1^A, \mathcal{T}\mathcal{M})$$

where  $\mathcal{P}_1^A$  is the enveloping category of  $A$  and  $\text{Fun}_{/\mathcal{M}}^{\mathcal{M}}(\mathcal{P}_1^A, \mathcal{T}\mathcal{M})$  denotes the category of  $\mathcal{M}$ -enriched lifts

$$\begin{array}{ccc} & & \mathcal{T}\mathcal{M} \\ & \nearrow \text{dotted} & \downarrow \\ \mathcal{P}_1^A & \xrightarrow[A]{} & \mathcal{M} \end{array}$$

of the underlying  $A$ -module  $A: \mathcal{P}_1^A \rightarrow \mathcal{M}$ .

Theorem 1.0.1, while pertaining to model categories, can also be used to obtain results in the  $\infty$ -categorical setting, using the rectification results of [PS14] and [NS15]. This is worked out in §4.4, where the following  $\infty$ -categorical analogue of the above result is established (see Theorem 4.4.3):

**THEOREM 1.0.3.** *Let  $\mathcal{C}$  be a closed symmetric monoidal, differentiable presentable  $\infty$ -category and let  $\mathcal{O}^\otimes = N^\otimes(\mathcal{P})$  be the operadic nerve a fibrant simplicial operad. Then the forgetful functor induces an equivalence of  $\infty$ -categories*

$$\mathcal{T}_A \text{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{T}_A \text{Mod}_A^{\mathcal{O}}(\mathcal{C}).$$

Here  $\text{Mod}_A^{\mathcal{O}}(\mathcal{C})$  is the  $\infty$ -category of  $A$ -modules in  $\mathcal{C}$ , which is closely related to the  $\infty$ -operad of  $A$ -modules defined in [Lur14, §3.3] (see Section 4.4). In the special case where  $\mathcal{C}$  is stable the conclusion of Theorem 1.0.3 reduces to the following result of Lurie [Lur14, Theorem 7.3.4.13]:

**COROLLARY 1.0.4.** *If, in addition to the above assumptions,  $\mathcal{C}$  is stable, then there is an equivalence of  $\infty$ -categories*

$$\mathcal{T}_A \text{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \text{Mod}_A^{\mathcal{O}}(\mathcal{C}).$$

While Theorem 1.0.3 is only applicable to  $\infty$ -operads which are nerves of simplicial operads (these are most likely all of them, see [CHH16],[HHM15]), it also applies to  $\infty$ -operads which are not necessarily unital or coherent, as is assumed in [Lur14, Theorem 7.3.4.13]. We also note that the model categorical statement Theorem 1.0.1 can handle not only simplicial operads, but also operads which are **enriched in  $\mathcal{M}$** . For example, this allows one to consider operads such as the Lie or Poisson operad, which do not come from simplicial operads, and thus are not covered by [Lur14, Theorem 7.3.4.13]. Such statements would most likely be translatable to an  $\infty$ -categorical language as soon as a suitable theory of enriched  $\infty$ -operads is set up.

This is a first in a series of papers concerned with the abstract cotangent complex formalism and its applications. In a subsequent paper [HNP16] we will further develop the aspects of the theory pertaining to Quillen cohomology and obstruction theory. We then use the main comparison results of this paper to study the tangent categories and Quillen cohomology of **enriched categories** and  **$\infty$ -categories**. For example, using Theorem 1.0.1 we obtain following corollary in loc. cit.:

**THEOREM 1.0.5.** *Let  $\mathcal{C}$  be an  $\infty$ -category. Then  $\mathcal{T}_{\mathcal{C}} \text{Cat}_{\infty}$  is naturally equivalent to the  $\infty$ -category of functors  $\text{Tw}(\mathcal{C}) \rightarrow \text{Spectra}$  from the twisted arrow category of  $\mathcal{C}$  to spectra.*

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## 2. Stabilization of model categories

The aim of this section is to associate to a – sufficiently nice – pointed model category  $\mathcal{M}$  a model category  $\mathrm{Sp}(\mathcal{M})$  of spectrum objects in  $\mathcal{M}$  which presents the universal stable  $\infty$ -category associated to the  $\infty$ -category underlying  $\mathcal{M}$ . When  $\mathcal{M}$  is a simplicial model category, one can use the suspension and loop functors induced by the simplicial (co)tensoring to give explicit models for spectrum objects in  $\mathcal{M}$  by means of Bousfield-Friedlander spectra or symmetric spectra (see [Hov01]). In non-simplicial contexts this can be done as soon as one chooses a Quillen adjunction realizing the loop-suspension adjunction. For the purposes of this paper it is desirable to have a uniform description of stabilization which does not depend on a simplicial structure or any other specific model for the loop-suspension adjunction. We will consequently follow a variant of the approach suggested by Heller in [Hel97], and describe spectrum objects in terms of  $(\mathbb{N} \times \mathbb{N})$ -diagrams (see also [Lur06, §8]).

Let us begin with an informal discussion outlining the approach. Let  $\mathcal{M}$  be a pointed model category, let  $X \in \mathcal{M}$  be a cofibrant object and  $Y \in \mathcal{M}$  a fibrant object. The key idea is that specifying a map of the form  $\Sigma X \rightarrow Y$ , or equivalently, a map  $X \rightarrow \Omega Y$ , is essentially equivalent to giving a commuting square

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Y \end{array} \quad (2.0.2)$$

in  $\mathcal{M}$  in which the objects  $Z$  and  $Z'$  are weakly contractible. The square (2.0.2) is homotopy coCartesian if and only if the corresponding map  $\Sigma X \rightarrow Y$  is an equivalence, and is homotopy Cartesian if and only if the adjoint map  $X \rightarrow \Omega Y$  is an equivalence. One can therefore describe pre-spectra as  $(\mathbb{N} \times \mathbb{N})$ -diagrams

$$\begin{array}{ccccc} X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \end{array}$$

in which all the off-diagonal entries are weakly contractible, so that the diagonal squares witness the structure maps of the pre-spectrum. Such a pre-spectrum is called a suspension spectrum if all diagonal squares

$$\begin{array}{ccc} X_{n,n} & \longrightarrow & X_{n,n+1} \\ \downarrow & & \downarrow \\ X_{n+1,n} & \longrightarrow & X_{n+1,n+1} \end{array} \quad (2.0.3)$$

are homotopy coCartesian and an  $\Omega$ -spectrum if all these squares are homotopy Cartesian.

## 2.1 The stable model structure

We shall now make the approach outlined above more precise. Recall that a **zero object** in a category is an object which is both initial and terminal and that a **weak zero object** in a model category  $\mathcal{M}$  is an object whose image in  $\text{Ho}(\mathcal{M})$  is a zero object. We will say that  $\mathcal{M}$  is strictly (resp. weakly) pointed if it admits a zero object (resp. weak zero object).

DEFINITION 2.1.1. Let  $\mathcal{M}$  be a weakly pointed model category. We will say that an  $(\mathbb{N} \times \mathbb{N})$ -diagram  $X_{\bullet, \bullet} : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{M}$  is

- (1) a **pre-spectrum** if all its off-diagonal entries are weak zero objects in  $\mathcal{M}$ ;
- (2) an  **$\Omega$ -spectrum** if it is a pre-spectrum and for each  $n \geq 0$ , the diagonal square (2.0.3) is homotopy Cartesian;
- (3) a **suspension spectrum** if it is a pre-spectrum and for each  $n \geq 0$ , the diagonal square (2.0.3) is homotopy coCartesian.

Remark 2.1.2. Let  $\iota : [1] \times [1] \rightarrow \mathbb{N} \times \mathbb{N}$  be the full inclusion of a square in  $\mathbb{N} \times \mathbb{N}$ . The functor  $\iota$  has the property that for any  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , the comma category  $\iota/(i, j)$  is either empty or has a terminal object. Using this, one easily checks that the left Kan extension functor  $\iota_! : \mathcal{M}^{[1] \times [1]} \rightarrow \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  preserves levelwise (trivial) cofibrations. Similarly, the comma categories  $(i, j)/\iota$  are either empty or have an initial object, from which it follows that the right Kan extension functor  $\iota_* : \mathcal{M}^{[1] \times [1]} \rightarrow \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  preserves levelwise (trivial) fibrations.

If  $X$  is injectively fibrant (resp. projectively cofibrant), it follows that each diagonal square (2.0.3) is an injectively fibrant (resp. projectively cofibrant) square in  $\mathcal{M}$ . Consequently, an injectively fibrant pre-spectrum  $X$  is an  $\Omega$ -spectrum if and only if the fibration

$$X_{n,n} \rightarrow X_{n+1,n} \times_{X_{n+1,n+1}} X_{n,n+1}$$

is a trivial fibration in  $\mathcal{M}$ . Dually, a projectively cofibrant pre-spectrum is a suspension spectrum if and only if each cofibration

$$X_{n+1,n} \coprod_{X_{n,n}} X_{n,n+1} \rightarrow X_{n+1,n+1}$$

is a trivial cofibration.

The notion of an  $\Omega$ -spectrum leads to a natural notion of a **stable equivalence**.

DEFINITION 2.1.3. Let  $\mathcal{M}$  be a weakly pointed combinatorial model category. We will say that a map  $f : X \rightarrow Y$  in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  is a **stable equivalence** if for every  $\Omega$ -spectrum  $Z$  the induced map

$$\text{Map}^h(Y, Z) \rightarrow \text{Map}^h(X, Z)$$

is a weak equivalence of spaces. Here the derived mapping spaces can be computed using either the projective or the injective model structure on  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$ , which are Quillen equivalent. We note that a stable equivalence between  $\Omega$ -spectra is always a levelwise equivalence

DEFINITION 2.1.4. Let  $\mathcal{M}$  be a weakly pointed combinatorial model category. The **projective stable model structure** on the category  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  – if it exists – is the model structure whose cofibrations are the projective cofibrations and whose weak equivalences are the stable equivalences. Similarly, the **injective stable model structure** on the category  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  is the model structure whose cofibrations are the levelwise cofibrations and whose weak equivalences are the stable equivalences. When they exist we will denote these model structures by  $\text{Sp}^{\text{proj}}(\mathcal{M})$  and  $\text{Sp}^{\text{inj}}(\mathcal{M})$  respectively. Omitting an explicit indication the notation  $\text{Sp}(\mathcal{M})$  will refer by default

to the **projective stable model structure**. We will refer to any of these model structures as the **stabilization** of  $\mathcal{M}$ .

The following lemma shows that the projective (res. injective) stable model structure, when it exists, is a left Bousfield localization of the projective (resp. injective) model structure on  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  at a concrete set of maps.

*Remark 2.1.5.* Since the derived mapping spaces in  $\mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}$  and  $\mathcal{M}_{\text{inj}}^{\mathbb{N} \times \mathbb{N}}$  coincide the question of whether an object in  $Z \in \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  is local with respect to a given map does not depend on whether we work in the injective or the projective model structure.

Because  $\mathcal{M}$  is combinatorial there exists a set  $\mathcal{D}$  of cofibrant objects of  $\mathcal{M}$  such that a map  $f : X \rightarrow Y$  in  $\mathcal{M}$  is a weak equivalence if and only if the induced map  $\text{Map}_{\mathcal{M}}^h(D, X) \rightarrow \text{Map}_{\mathcal{M}}^h(D, Y)$  is a weak equivalence of spaces for every  $D \in \mathcal{D}$  (see e.g. [Dug01, Proposition 4.7]). For  $(n, m) \in \mathbb{N} \times \mathbb{N}$  let us denote by  $h_{n,m} = \text{hom}((m, n), -) : \mathbb{N} \times \mathbb{N} \rightarrow \text{Set}$  the associated corepresentable functor. We will denote by  $\otimes$  the natural tensoring of  $\mathcal{M}$  over sets.

**LEMMA 2.1.6.** *Let  $\mathcal{M}$  be a weakly pointed combinatorial model category and let  $\mathcal{D}$  be a set of objects as above. Let  $Z \in \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  be an object. Then the following statements hold:*

(1)  *$Z$  is a pre-spectrum if and only if  $Z$  is local with respect to the set of maps*

$$(*) \quad \emptyset \rightarrow h_{n,m} \otimes D$$

*for every  $D \in \mathcal{D}$  and  $n \neq m$ ,*

(2)  *$Z$  is an  $\Omega$ -spectrum if and only if it is a pre-spectrum which is furthermore local with respect to the set of maps*

$$(**) \quad \left[ h_{n+1,n} \coprod_{h_{n+1,n+1}} h_{n,n+1} \right] \otimes D \rightarrow h_{n,n} \otimes D$$

*for every  $D \in \mathcal{D}$  and every  $n \geq 0$ .*

*Proof.* The association  $Z \mapsto Z_{m,n}$  is a right Quillen functor  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathcal{M}$  (with respect either the injective or the projective model structure) whose left adjoint sends  $A$  to  $h_{n,m} \otimes A$ . By adjunction  $Z$  is local with respect to the first set of maps if and only if each  $Z_{m,n} \in \mathcal{M}$  with  $m \neq n$  is local with respect to  $\emptyset \rightarrow D$  for every  $D$ . By the characteristic property of  $\mathcal{D}$  (and since  $\mathcal{M}$  is weakly pointed) this is the same as saying that  $Z_{m,n}$  is a weak zero object.

To prove (2), consider the functor  $\mathcal{R}_n : \mathcal{M}_{\text{inj}}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathcal{M}$  given by  $\mathcal{R}_n(Z) = Z_{n+1,n} \times_{Z_{n+1,n+1}} Z_{n,n+1}$ . This is a right Quillen functor whose left adjoint  $\mathcal{L}_n : \mathcal{M} \rightarrow \mathcal{M}_{\text{inj}}^{\mathbb{N} \times \mathbb{N}}$  sends an object  $A \in \mathcal{M}$  to the diagram  $[h_{n+1,n} \coprod_{h_{n+1,n+1}} h_{n,n+1}] \otimes A$  (indeed, the latter clearly sends cofibrations trivial cofibrations to injective cofibrations and trivial cofibrations, respectively). Using adjunction and Remark 2.1.2 we get

$$\text{Map}^h(\mathcal{L}_n(A^{\text{cof}}), Z) \simeq \text{Map}^h(A, \mathcal{R}_n(Z^{\text{fib}})) \simeq \text{Map}^h(A, Z_{n,n+1} \times_{Z_{n+1,n+1}}^h Z_{n+1,n})$$

where  $Z \rightarrow Z^{\text{fib}}$  is an injective fibrant replacement of  $Z$ . Now the characteristic property of  $\mathcal{D}$  shows that  $Z$  is local with respect to all maps  $[h_{n+1,n} \coprod_{h_{n+1,n+1}} h_{n,n+1}] \otimes D \rightarrow h_{n,n} \otimes D$  if and only if the natural map

$$Z_{n,n} \rightarrow Z_{n,n+1} \times_{Z_{n+1,n+1}}^h Z_{n+1,n}$$

is a weak equivalence, i.e., if and only if the (2.0.3) is homotopy Cartesian.  $\square$



COROLLARY 2.1.7. *Let  $\mathcal{M}$  be a weakly pointed combinatorial model category.*

- (1) *The projective (resp. injective) stable model structure exists if and only if the left Bousfield localization of the projective (resp. injective) model structure on  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  with respect to the maps  $(*)$  and  $(**)$  exists, in which case they coincide.*
- (2) *If  $\mathcal{M}$  is left proper then both the injective and the projective stable model structures on  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  exist, and the identity adjunction is a Quillen equivalence  $\mathrm{Sp}^{\mathrm{proj}}(\mathcal{M}) \xrightarrow[\perp]{\simeq} \mathrm{Sp}^{\mathrm{inj}}(\mathcal{M})$ .*
- (3) *If either the projective or the injective stabilization exists, then there is a Quillen adjunction*

$$\Sigma^\infty : \mathcal{M} \xrightleftharpoons[\perp]{} \mathrm{Sp}(\mathcal{M}) : \Omega^\infty$$

where  $\Sigma^\infty(X)$  is the constant diagram with value  $X$  and  $\Omega^\infty X_{\bullet\bullet} = X_{0,0}$ .

*Proof.* Claim (1) is an immediate consequence of Lemma 2.1.6. Claim (2) follows from (1) and the classical existence theorems of left Bousfield localizations (see [Hir03]). For Claim (3) it is enough to check that  $\Omega^\infty$  is a right Quillen functor with respect to the projective model structure on  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$ , which is clear.  $\square$

*Remark 2.1.8.* For any object  $X \in \mathcal{M}$  we call  $\Sigma^\infty X$  the **suspension spectrum** of  $X$ . At the moment  $\Sigma^\infty X$  does not resemble the classical notion of a suspension spectrum and is not even a suspension spectrum in the sense of Definition 2.1.1. We will show in Lemma 2.3.1 that  $\Sigma^\infty X$  is stably equivalent to a suspension spectrum whose zeroth object is  $X$ .

*Remark 2.1.9.* When the projective (resp. injective) stable model structure does not exist, the class of projective (resp. injective) cofibrations which are also stable equivalences is not closed under pushouts. However, this class is closed under pushouts along maps with a levelwise cofibrant domains and codomains (indeed, such pushouts are always homotopy pushouts  $\mathcal{M}_{\mathrm{inj}}^{\mathbb{N} \times \mathbb{N}}$  and hence in  $\mathcal{M}_{\mathrm{proj}}^{\mathbb{N} \times \mathbb{N}}$  as well).

*Remark 2.1.10.* Any levelwise cofibrant object  $X \in \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  admits a stable equivalence  $X \rightarrow E$  to an  $\Omega$ -spectrum. This either follows formally from inspecting the proof of the existence of Bousfield localizations in the left proper case, or – under some mild restrictions – from the explicit constructions in Remark 2.3.4 and Corollary 2.4.6.

PROPOSITION 2.1.11. *Any Quillen pair  $\mathcal{L} : \mathcal{M} \xrightleftharpoons[\perp]{} \mathcal{N} : \mathcal{R}$  between weakly pointed left proper combinatorial model categories fits into a commuting diagram of Quillen adjunctions*

$$\begin{array}{ccc} \mathrm{Sp}(\mathcal{M}) & \xrightleftharpoons[\perp]{\mathrm{Sp}(\mathcal{L})} & \mathrm{Sp}(\mathcal{N}) \\ \Sigma^\infty \uparrow \downarrow \Omega^\infty & \mathrm{Sp}(\mathcal{R}) & \Sigma^\infty \uparrow \downarrow \Omega^\infty \\ \mathcal{M} & \xrightleftharpoons[\mathcal{R}]{\mathcal{L}} & \mathcal{N} \end{array}$$

where the top horizontal Quillen pair is a Quillen equivalence if  $\mathcal{L} \dashv \mathcal{R}$  is such.

*Proof.* Clearly  $\mathcal{L} \vdash \mathcal{R}$  induces a Quillen pair  $\mathcal{L}^{\mathbb{N} \times \mathbb{N}} : \mathcal{M}_{\mathrm{proj}}^{\mathbb{N} \times \mathbb{N}} \xrightleftharpoons[\perp]{} \mathcal{N}_{\mathrm{proj}}^{\mathbb{N} \times \mathbb{N}} : \mathcal{R}^{\mathbb{N} \times \mathbb{N}}$  between the respective projective model structures, which is a Quillen equivalence if  $\mathcal{L} \vdash \mathcal{R}$  is such. By standard Bousfield localization techniques ([Hir03, Section 3], see also Lemma 3.1.2 below), this Quillen adjunction descends to the stable model structure because  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}$  preserves  $\Omega$ -spectra. Furthermore, when  $\mathcal{L} \vdash \mathcal{R}$  is a Quillen equivalence  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}$  also detects  $\Omega$ -spectra, and the induced Quillen adjunction between the stable model structures is a Quillen equivalence. The compatibility with  $\Sigma^\infty$  and  $\Omega^\infty$  is immediate from the definitions.  $\square$

*Remark 2.1.12.* The statement of Proposition 2.1.11 holds for the injective stable model structures as well, with the exact same proof.

## 2.2 The $\infty$ -categorical stabilization

Recall that any model category  $\mathcal{M}$  (and in fact any relative category) has a canonically associated  $\infty$ -category  $\mathcal{M}_\infty$ , obtained by formally inverting the weak equivalences of  $\mathcal{M}$  (see e.g. [Hin13] for a thorough account, or alternatively, the discussion in [BHH16, §2.2]). Furthermore, a Quillen adjunction  $\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$  induces an adjunction of  $\infty$ -categories  $\mathcal{L}_\infty : \mathcal{M}_\infty \rightleftarrows \mathcal{N}_\infty : \mathcal{R}_\infty$  ([Hin13, Proposition 1.5.1]).

**DEFINITION 2.2.1.** A model category  $\mathcal{M}$  is a **stable** model category if  $\mathcal{M}_\infty$  is stable in the sense of [Lur14], i.e. if  $\mathcal{M}_\infty$  is pointed and the adjunction of  $\infty$ -categories  $\Sigma : \mathcal{M}_\infty \rightleftarrows \mathcal{M}_\infty : \Omega$  is an adjoint equivalence.

*Remark 2.2.2.* Alternatively, one can characterize the stable model categories as those weakly pointed model categories in which a square is homotopy Cartesian if and only if it is homotopy coCartesian (see [Lur14]).

*Remark 2.2.3.* An adjunction between  $\infty$ -categories is an equivalence if and only if the induced adjunction on homotopy categories is an equivalence (indeed, both statements just depend on whether the unit and counit are equivalences). Consequently, a model category  $\mathcal{M}$  is stable if and only if it is weakly pointed and  $\Sigma : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \Omega$  is an equivalence of categories (cf. [Hov01]).

Our next goal is to show that the  $\infty$ -category associated to the stabilization  $\mathrm{Sp}(\mathcal{M})$  of a model category  $\mathcal{M}$  presents the universal stable  $\infty$ -category associated to  $\mathcal{M}_\infty$ , in the sense of [Lur14, Proposition 1.4.2.22]. For this it will be useful to consider the operation of stabilization in the not-necessarily pointed setting. Recall that if  $\mathcal{C}$  is a presentable  $\infty$ -category then the  $\infty$ -category  $\mathcal{C}_* \stackrel{\mathrm{def}}{=} \mathcal{C}_{*/}$  of objects under the terminal object is the universal pointed presentable  $\infty$ -category receiving a colimit preserving functor from  $\mathcal{C}$ . Since any stable  $\infty$ -category is necessarily pointed we see that any colimit preserving functor from  $\mathcal{C}$  to a stable presentable  $\infty$ -category factors uniquely through  $\mathcal{C}_*$ . The composition  $\mathcal{C} \longrightarrow \mathcal{C}_* \longrightarrow \mathrm{Sp}(\mathcal{C}_*)$  thus exhibits  $\mathrm{Sp}(\mathcal{C}_*)$  as the universal stable presentable  $\infty$ -category admitting a colimit preserving functor from  $\mathcal{C}$ . Given a left proper combinatorial model category  $\mathcal{M}$  we will therefore consider  $\mathrm{Sp}(\mathcal{M}_*)$  also as the stabilization of  $\mathcal{M}$ , where  $\mathcal{M}_* = \mathcal{M}_{*/}$  is equipped with the coslice model structure. We note that when  $\mathcal{M}$  is already weakly pointed we have a Quillen equivalence  $\mathcal{M}_* \xrightarrow[\simeq]{\simeq} \mathcal{M}$  and so this poses no essential ambiguity. We will denote by  $\Sigma_+^\infty : \mathcal{M} \rightleftarrows \mathrm{Sp}(\mathcal{M}_*) : \Omega_+^\infty$  the composition of Quillen adjunctions

$$\Sigma_+^\infty : \mathcal{M} \xrightleftharpoons[u]{(-)\amalg *} \mathcal{M}_* \xrightleftharpoons[\Omega^\infty]{\Sigma^\infty} \mathrm{Sp}(\mathcal{M}_*) : \Omega_+^\infty .$$

We note that the above construction is only appropriate if  $\mathcal{M}_*$  is actually a model for the  $\infty$ -category  $(\mathcal{M}_\infty)_*$ . We shall begin by addressing this issue.

**LEMMA 2.2.4.** *Let  $\mathcal{M}$  be a combinatorial model category and  $X \in \mathcal{M}$  an object. Assume either that  $X$  is cofibrant or that  $\mathcal{M}$  is left proper. Then the natural functor of  $\infty$ -categories  $(\mathcal{M}_{X/})_\infty \longrightarrow (\mathcal{M}_\infty)_{X/}$  is an equivalence.*

*Proof.* If  $\mathcal{M}$  is left proper then any weak equivalence  $f : X \longrightarrow X'$  induces a Quillen equivalence  $f_! : \mathcal{M}_{X/} \rightleftarrows \mathcal{M}_{X'/} : f^*$  and hence an equivalence between the associated  $\infty$ -categories. Similarly,

for any model category the adjunction  $f_! \dashv f^*$  is a Quillen equivalence when  $f$  is a weak equivalence between cofibrant objects. It therefore suffices to prove the lemma under the assumption that  $X$  is fibrant-cofibrant.

Note that for any Quillen equivalence  $\mathcal{L} : \mathcal{N} \xrightarrow[\mathcal{L}]{\mathcal{L}} \mathcal{M} : \mathcal{R}$  and a fibrant object  $X \in \mathcal{M}$ , the induced Quillen pair  $\mathcal{N}_{\mathcal{R}(X)/} \xrightarrow[\mathcal{L}]{\mathcal{L}} \mathcal{M}_{X/}$  is a Quillen equivalence as well (see Construction 3.1.4 and Remark 3.1.6). By [Dug01] there exists a simplicial, left proper combinatorial model category  $\mathcal{M}'$ , together with a Quillen equivalence  $\mathcal{M}' \xrightarrow[\mathcal{L}]{\mathcal{L}} \mathcal{M}$ . We may therefore reduce to the case where  $\mathcal{M}$  is furthermore simplicial and  $X \in \mathcal{M}$  is fibrant-cofibrant, in which case the result follows from [Lur09, Lemma 6.1.3.13].  $\square$

**PROPOSITION 2.2.5.** *Let  $\mathcal{M}$  be a left proper combinatorial model category. Then the functor  $(\Omega_+^\infty)_\infty : \mathrm{Sp}(\mathcal{M}_*)_\infty \rightarrow \mathcal{M}_\infty$  exhibits  $\mathrm{Sp}(\mathcal{M}_*)_\infty$  as the stabilization of  $\mathcal{M}_\infty$  (in the sense of the universal property of [Lur14, Proposition 1.4.2.23]).*

*Proof.* Since  $\mathcal{M}$  is left proper, Lemma 2.2.4 implies that the natural functor  $(\mathcal{M}_*)_\infty \rightarrow (\mathcal{M}_\infty)_*$  is an equivalence. It therefore suffices to show that for a weakly pointed model category  $\mathcal{M}$ , the map  $(\Omega^\infty)_\infty : \mathrm{Sp}(\mathcal{M})_\infty \rightarrow \mathcal{M}_\infty$  exhibits  $\mathrm{Sp}(\mathcal{M})_\infty$  as the stabilization of the pointed  $\infty$ -category  $\mathcal{M}_\infty$ .

Since  $\mathrm{Sp}(\mathcal{M})$  is a left Bousfield localization of  $\mathcal{M}_{\mathrm{proj}}^{\mathbb{N} \times \mathbb{N}}$  (Corollary 2.1.7) it follows that the underlying  $\infty$ -category  $\mathrm{Sp}(\mathcal{M})_\infty$  is equivalent to the full subcategory of  $(\mathcal{M}_{\mathrm{proj}}^{\mathbb{N} \times \mathbb{N}})_\infty$  spanned by the local objects, i.e., by the  $\Omega$ -spectra. By [Lur09, Proposition 4.2.4.4] the natural map

$$(\mathcal{M}^{\mathbb{N} \times \mathbb{N}})_\infty \rightarrow (\mathcal{M}_\infty)^{\mathbb{N} \times \mathbb{N}}$$

is an equivalence of  $\infty$ -categories. We may therefore conclude that  $\mathrm{Sp}(\mathcal{M})_\infty$  is equivalent to the full subcategory  $\mathrm{Sp}'(\mathcal{M}_\infty) \subseteq (\mathcal{M}_\infty)^{\mathbb{N} \times \mathbb{N}}$  spanned by those diagrams  $\mathcal{F} : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{M}_\infty$  such that  $\mathcal{F}(n, m)$  is zero object for  $n \neq m$  and  $\mathcal{F}$  restricted to each diagonal square is Cartesian. We now claim that the evaluation at  $(0, 0)$  functor  $\mathrm{ev}_{(0,0)} : \mathrm{Sp}'(\mathcal{M}_\infty) \rightarrow \mathcal{M}_\infty$  exhibits  $\mathrm{Sp}'(\mathcal{M}_\infty)$  as the stabilization of  $\mathcal{M}_\infty$ . By [Lur14, Proposition 1.4.2.24] it will suffice to show that  $\mathrm{ev}_{(0,0)}$  lifts to an equivalence between  $\mathrm{Sp}'(\mathcal{M}_\infty)$  and the homotopy limit of the tower

$$\cdots \rightarrow \mathcal{M}_\infty \xrightarrow{\Omega} \mathcal{M}_\infty \xrightarrow{\Omega} \mathcal{M}_\infty \quad (2.2.1)$$

The proof of this fact is completely analogous to the proof of [Lur06, Proposition 8.14]. Indeed, one may consider for each  $n$  the  $\infty$ -category  $\mathcal{D}'_n$  of  $(\mathbb{N}_{\leq n} \times \mathbb{N}_{\leq n})$ -diagrams in  $\mathcal{M}_\infty$  which are contractible off-diagonal and have Cartesian squares on the diagonal. It follows from Lemma 8.12 and Lemma 8.13 of [Lur06] (as well as [Lur09, Proposition 4.3.2.15]) that the functor  $\mathrm{ev}_{(n,n)} : \mathcal{D}'_n \rightarrow \mathcal{M}_\infty$  is a trivial Kan fibration (hence a categorical equivalence). Under these equivalences, the restriction functor  $\mathcal{D}'_{n+1} \rightarrow \mathcal{D}'_n$  is identified with the loop functor  $\Omega : \mathcal{M}_\infty \rightarrow \mathcal{M}_\infty$ . It follows that the homotopy limit of the tower 2.2.1 can be identified with the homotopy limit of the tower of restriction functors  $\{\cdots \rightarrow \mathcal{D}'_2 \rightarrow \mathcal{D}'_1 \rightarrow \mathcal{D}'_0\}$ . Since these restriction functors are categorical fibrations between  $\infty$ -categories, the homotopy limit agrees with the actual limit, which is the  $\infty$ -category  $\mathrm{Sp}'(\mathcal{M}_\infty)$ .  $\square$

The next two corollaries follow immediately from the universal property of the  $\infty$ -categorical stabilization:

**COROLLARY 2.2.6.** *Let  $\mathcal{M}$  be a weakly pointed left proper combinatorial model category. Then  $\mathrm{Sp}(\mathcal{M})$  is a stable model category.*

**COROLLARY 2.2.7.** *If  $\mathcal{M}$  is a stable model category, then the adjunction  $\Sigma^\infty \dashv \Omega^\infty$  of Corollary 2.1.7 is a Quillen equivalence.*

*Remark 2.2.8.* Instead of invoking Proposition 2.2.5 to establish Corollary 2.2.6, one can prove directly at the model-categorical level that  $\Omega : \mathrm{Ho}(\mathrm{Sp}(\mathcal{M})) \rightarrow \mathrm{Ho}(\mathrm{Sp}(\mathcal{M}))$  is an equivalence. For this, it will be convenient to make use of the **shift functors**

$$[-n] : \mathrm{Sp}(\mathcal{M}) \rightleftarrows \mathrm{Sp}(\mathcal{M}) : [n] \quad n \geq 0$$

given by  $X[n]_{\bullet\bullet} := X_{\bullet+n, \bullet+n}$  and  $X[-n]_{\bullet\bullet} = X_{\bullet-n, \bullet-n}$  (where  $X_{i,j} = \emptyset$  when  $i < 0$  or  $j < 0$ ). These form a Quillen pair since the functor  $[-n]$  preserves levelwise weak equivalences and cofibrations, while  $[n]$  preserves  $\Omega$ -spectra. One can now easily check that  $\Omega \circ \mathbb{R}[1]$  and  $\mathbb{R}[1] \circ \mathbb{L}[-1]$  are both naturally isomorphic to the identity functor on  $\mathrm{Ho}(\mathrm{Sp}(\mathcal{M}))$ , from which the result follows.

*Remark 2.2.9.* Even if  $\mathcal{M}$  is a (weakly pointed, combinatorial) model category which is not left proper we can still consider the full relative subcategory  $\mathrm{Sp}'(\mathcal{M}) \subseteq \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  spanned by  $\Omega$ -spectra (with weak equivalences the levelwise weak equivalences). The composite functor  $\mathrm{Sp}'(\mathcal{M})_\infty \rightarrow (\mathcal{M}^{\mathbb{N} \times \mathbb{N}})_\infty \xrightarrow{\sim} (\mathcal{M}_\infty)^{\mathbb{N} \times \mathbb{N}}$  identifies  $\mathrm{Sp}'(\mathcal{M})_\infty$  with the full sub- $\infty$ -category  $\mathrm{Sp}'(\mathcal{M}_\infty) \subseteq (\mathcal{M}_\infty)^{\mathbb{N} \times \mathbb{N}}$  spanned by those diagrams which are contractible off diagonal and have Cartesian diagonal squares. The proof of Proposition 2.2.5 now implies that for **any** weakly pointed combinatorial model category  $\mathcal{M}$ , the stabilization of  $\mathcal{M}_\infty$  can be modeled by the relative category  $\mathrm{Sp}'(\mathcal{M})$ .

*Remark 2.2.10.* One may also model the stabilization of  $\mathcal{M}_\infty$  by the relative category  $\mathrm{Sp}''(\mathcal{M}) \subseteq \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  spanned by **pre-spectrum** objects and stable weak equivalences between them. This follows from the fact that one can functorially replace a levelwise cofibrant pre-spectrum  $X$  by an  $\Omega$ -spectrum  $X^\Omega$  equipped with a stable equivalence  $X \rightarrow X^\Omega$  (see Remark 2.1.10) and the fact that a stable weak equivalence between  $\Omega$ -spectra is a levelwise equivalence. Of course, when the stable model structure exists this is just a direct corollary of the fact that every object in  $\mathrm{Sp}(\mathcal{M})$  is stably equivalent to a pre-spectrum (see Remark 2.3.4 below).

*Remark 2.2.11.* By [Rob12, Proposition 4.15] the stabilization of model categories via Bousfield-Friedlander spectra ([Hov01]) is also a model for the  $\infty$ -categorical stabilization. Since both are combinatorial model categories, any equivalence of the underlying  $\infty$ -categories can be expressed as a zig-zag of Quillen equivalences (see [Lur09, Remark A.3.7.7]). In particular, for a strictly pointed, left proper, simplicial combinatorial model category  $\mathcal{M}$ , the stabilization  $\mathrm{Sp}(\mathcal{M})$  is Quillen equivalent to the model category of Bousfield-Friedlander spectra in  $\mathcal{M}$ .

*Remark 2.2.12.* Another closely related model is that of **reduced excisive functors** (see e.g. [Lyd98]). Let  $\mathcal{S}_*^{\mathrm{fin}}$  denote the relative category of finite simplicial sets. When  $\mathcal{M}$  is left proper and combinatorial we may form the left Bousfield localization  $\mathrm{Exc}(\mathcal{M})$  of the projective model structure on  $\mathcal{M}^{\mathcal{S}_*^{\mathrm{fin}}}$  in which the local objects are the relative reduced excisive functors. Restriction along  $\iota : \{S^0\} \hookrightarrow \mathcal{S}_*^{\mathrm{fin}}$  then yields a right Quillen functor  $\iota^* : \mathrm{Exc}(\mathcal{M}) \rightarrow \mathcal{M}$ . Using [Lur09, Proposition 4.2.4.4] and [Lur14, §1.4.2] one may then show that the induced functor  $\iota_\infty^* : \mathrm{Exc}(\mathcal{M})_\infty \rightarrow \mathcal{M}_\infty$  exhibits  $\mathrm{Exc}(\mathcal{M})_\infty$  as the stabilization of  $\mathcal{M}_\infty$ . In this case one can even construct a direct Quillen equivalence between  $\mathrm{Sp}(\mathcal{M})$  and  $\mathrm{Exc}(\mathcal{M})$ . Indeed, let  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{S}_*^{\mathrm{fin}}$  be a suspension spectrum object such that  $f(0,0) \cong S^0$ . Then restriction along  $f$  determines a right Quillen functor  $f^* : \mathrm{Exc}(\mathcal{M}) \rightarrow \mathrm{Sp}(\mathcal{M})$  such that  $\Omega^\infty \circ f^* \cong \iota^*$ . To show that  $f^*$  is a right Quillen equivalence it is enough to check that the induced functor  $f_\infty^* : \mathrm{Exc}(\mathcal{M})_\infty \rightarrow \mathrm{Sp}(\mathcal{M})_\infty$  is an equivalence of  $\infty$ -categories. But this now follows formally from the universal property shared by both sides.

### 2.3 Suspension spectra

The canonical adjunction  $\Sigma^\infty : \mathcal{M} \rightleftarrows \mathrm{Sp}(\mathcal{M}) : \Omega^\infty$  appearing in Corollary 2.1.7 is a model for the classical “suspension-infinity/loop-infinity” adjunction. This might seem surprising at first sight as the object  $\Sigma^\infty(X)$  is by definition a **constant**  $(\mathbb{N} \times \mathbb{N})$ -**diagram**, and not a suspension spectrum. In this section we will prove a convenient replacement lemma showing that up to a stable equivalence every constant spectrum object can be replaced with a suspension spectrum, which is unique in a suitable sense (see Remark 2.3.2). This can be used, for example, in order to functorially replace  $\Sigma^\infty(X)$  with a suspension spectrum, whenever the need arises (see Corollary 2.3.3 below). While mostly serving for intuition purposes in this paper, Lemma 2.3.1 is also intended for more direct applications in a subsequent paper [HNP16].

**LEMMA 2.3.1.** *Let  $\mathcal{M}$  be a combinatorial model category. Let  $f : X \rightarrow Y$  be a map in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  such that  $X$  is constant and levelwise cofibrant and  $Y$  is a suspension spectrum. Then there exists a factorization  $X \xrightarrow{f'} X' \xrightarrow{f''} Y$  of  $f$  such that  $X'$  is a suspension spectrum,  $f'$  is a stable equivalence and  $f'_{0,0} : X_{0,0} \rightarrow X'_{0,0}$  is a weak equivalence. In particular, if  $f_{0,0} : X_{0,0} \rightarrow Y_{0,0}$  is already a weak equivalence then  $f$  is a stable equivalence.*

*Proof.* Let us say that an object  $Z_{\bullet\bullet} \in \mathrm{Sp}(\mathcal{M})$  is a suspension spectrum up to  $n$  if  $Z_{m,k}$  is weakly contractible whenever  $m \neq k$  and  $\min(m,k) < n$  and if the  $m$ 'th diagonal square is a pushout square for  $m < n$ . In particular, the condition of being a suspension spectrum up to 0 is vacuous. We will now construct a sequence of levelwise cofibrations and stable equivalences

$$X = P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_n \rightarrow P_{n+1} \rightarrow \cdots$$

over  $Y$  such that each  $P_n$  is a levelwise cofibrant suspension spectrum up to  $n$  and the map  $(P_n)_{m,k} \rightarrow (P_{n+1})_{m,k}$  is an isomorphism whenever  $\min(m,k) < n$  or  $m = k = n$ . Then  $X' \stackrel{\mathrm{def}}{=} \mathrm{colim}_n P_n \simeq \mathrm{hocolim}_n P_n$  is a suspension spectrum by construction and the map  $f : X \rightarrow X'$  satisfies the required conditions (see Remark 2.1.9).

Given a cofibrant object  $Z \in \mathcal{M}$  equipped with a map  $Z \rightarrow Y_{n,n}$ , let us denote the cone of the composed map  $Z \rightarrow Y_{n,n} \rightarrow Y_{n,n+1}$  by  $Z \rightarrow C_{n,n+1}(Z) \rightarrow Y_{n,n+1}$  and the cone of the map  $Z \rightarrow Y_{n,n} \rightarrow Y_{n+1,n}$  by  $Z \rightarrow C_{n+1,n}(Z) \rightarrow Y_{n+1,n}$ . Since  $Y$  is weakly contractible off diagonal it follows that  $C_{n,n+1}(Z)$  and  $C_{n+1,n}(Z)$  are weak zero objects. Let  $\Sigma_Y(Z) := C_{n,n+1}(Z) \amalg_Z C_{n+1,n}(Z)$  be the induced model for the suspension of  $Z$  in  $\mathcal{M}$ . By construction the object  $\Sigma_Y(Z)$  carries a natural map  $\Sigma_Y(Z) \rightarrow Y_{n+1,n+1}$ . Let us now define  $Q_{n,n+1}(Z)$ ,  $Q_{n+1,n}(Z)$  and  $Q_{n+1}(Z)$  by forming the following diagram in  $\mathcal{M}_{/Y}^{\mathbb{N} \times \mathbb{N}}$ :

$$\begin{array}{ccccc} & & h_{n,n+1} \otimes Z & \longrightarrow & h_{n,n+1} \otimes C_{n,n+1}(Z) \\ & & \downarrow & & \downarrow \\ h_{n+1,n} \otimes Z & \longrightarrow & h_{n,n} \otimes Z & \longrightarrow & Q_{n,n+1}(Z) \\ \downarrow & & \downarrow & & \downarrow \\ h_{n+1,n} \otimes C_{n+1,n}(Z) & \longrightarrow & Q_{n+1,n}(Z) & \longrightarrow & Q_{n+1}(Z). \end{array}$$

Since all objects in this diagram are levelwisewise cofibrant and the top right horizontal map is a levelwise cofibration and a stable equivalence, all the right horizontal maps are levelwise cofibrations and stable equivalences (see Remark 2.1.9). Similarly, since the left bottom vertical map is a levelwise cofibration and a stable equivalence the same holds for all bottom vertical maps. It then follows that  $h_{n,n} \otimes Z \rightarrow Q_{n+1}(Z)$  is a levelwise cofibration and a stable equivalence

over  $Y$ . We note that by construction the shifted diagram  $Q_{n+1}(Z)[n+1]$  is constant on  $\Sigma_Y(Z)$  (see Remark 2.2.8).

Let us now assume that we have constructed  $P_n \rightarrow Y$  such that  $P_n$  is a suspension spectrum up to  $n$  and such that the shifted object  $P_n[n]$  (Remark 2.2.8) is a constant diagram. This is clearly satisfied by  $P_0 \stackrel{\text{def}}{=} X$ . We now define  $P_{n+1}$  inductively as the pushout

$$\begin{array}{ccc} h_{n,n} \otimes (P_n)_{n,n} & \longrightarrow & Q_{n+1}((P_n)_{n,n}) \\ \downarrow & & \downarrow \lrcorner \\ P_n & \longrightarrow & P_{n+1} \end{array}$$

Since the left vertical map becomes an isomorphism after applying the shift  $[n]$ , so does the right vertical map in the above square. It follows that  $P_{n+1}[n+1]$  is constant and that the  $n$ 'th diagonal square of  $P_{n+1}$  is homotopy coCartesian by construction. This means that  $P_{n+1}$  is a suspension spectrum up to  $n$ . Furthermore, by construction the map  $P_n \rightarrow P_{n+1}$  is a levelwise cofibration and a stable equivalence which is an isomorphism at  $(m, k)$  whenever at least one of  $m, k$  is smaller than  $n$  or  $k = m = n$ .  $\square$

*Remark 2.3.2.* Given an injective cofibrant constant spectrum object  $X$ , Lemma 2.3.1 provides a stable equivalence  $X \rightarrow X'$  from  $X$  to a suspension spectrum which induces an equivalence in degree  $0, 0$ . These “suspension spectrum replacements” can be organized into a category, and Lemma 2.3.1 can be used to show that the nerve of this category is weakly contractible. We may hence consider a suspension spectrum replacement in the above sense as essentially unique.

**COROLLARY 2.3.3.** *Let  $X \in \mathcal{M}$  be a cofibrant object. Then there exists a stable equivalence  $\Sigma^\infty X \rightarrow \overline{\Sigma}^\infty \overline{X}$  whose codomain is a suspension spectrum. Furthermore,  $\overline{\Sigma}^\infty \overline{X}$  can be chosen to depend functorially on  $X$  and  $\overline{\Sigma}^\infty \overline{X}_{0,0} \cong X$ .*

*Remark 2.3.4.* A similar but simpler construction replaces any levelwise cofibrant  $(\mathbb{N} \times \mathbb{N})$ -diagram  $X$  by a weakly equivalent pre-spectrum: let  $X^{(0)} = X$  and inductively define  $X^{(k+1)}$  such that  $X^{(k)} \rightarrow X^{(k+1)}$  is a pushout along

$$\coprod_{m+n=k, m \neq n} h_{m,n} \otimes X_{m,n}^{(k)} \longrightarrow \coprod_{m+n=k, m \neq n} h_{m,n} \otimes C(X_{m,n}^{(k)}).$$

The map  $X^{(k)} \rightarrow X^{(k+1)}$  is then an isomorphism below the line  $m+n=k$  and replaces the off-diagonal entries on that line by their cones. It is a levelwise cofibration and a stable equivalence, being the pushout of such a map with cofibrant target (see Remark 2.1.9). The (homotopy) colimit of the resulting sequence of stable equivalences yields the desired pre-spectrum replacement.

## 2.4 Differentiable model categories and $\Omega$ -spectra

Our goal in this subsection is to give a description of the fibrant replacement of a pre-spectrum, which resembles the classical fibrant replacement of spectra (see [Hov01], or [Lur06, Corollary 8.17] for the  $\infty$ -categorical analogue). This description requires some additional assumptions on the model category at hand, which we first spell out.

Let  $f : \mathcal{J} \rightarrow \mathcal{M}$  be a diagram in a combinatorial model category  $\mathcal{M}$ . Recall that a cocone  $\overline{f} : \mathcal{J}^\triangleright \rightarrow \mathcal{M}$  over  $f$  is called a **homotopy colimit diagram** if for some projectively cofibrant replacement  $f^{\text{cof}} \rightarrow f$ , the composed map  $\text{colim } f^{\text{cof}}(i) \rightarrow \text{colim } f(i) \rightarrow \overline{f}(\ast)$  is a weak equivalence (where  $\ast \in \mathcal{J}^\triangleright$  denotes the cone point). A functor  $\mathcal{G} : \mathcal{M} \rightarrow \mathcal{N}$  preserving weak equivalences is said to **preserve  $\mathcal{J}$ -indexed homotopy colimits** if it maps  $\mathcal{J}^\triangleright$ -indexed homotopy colimit diagrams to homotopy colimit diagrams.

DEFINITION 2.4.1 cf. [Lur14, Definition 6.1.1.6]. Let  $\mathcal{M}$  be a model category and let  $\mathbb{N}$  be the poset of non-negative integers as above. We will say that  $\mathcal{M}$  is **differentiable** if for every finite category  $\mathcal{J}$  the right derived limit functor  $\mathbb{R}\lim : \mathcal{M}^{\mathcal{J}} \rightarrow \mathcal{M}$  preserves  $\mathbb{N}$ -indexed homotopy colimits. We will say that a Quillen adjunction  $\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$  is **differentiable** if  $\mathcal{M}$  and  $\mathcal{N}$  is differentiable and  $\mathbb{R}\mathcal{R}$  preserves  $\mathbb{N}$ -indexed homotopy colimits.

Remark 2.4.2. The condition that  $\mathcal{M}$  be differentiable can be equivalently phrased by saying that the derived colimit functor  $\mathbb{L}\mathrm{colim} : \mathcal{M}^{\mathbb{N}} \rightarrow \mathcal{M}$  preserves finite homotopy limits. This means, in particular, that if  $\mathcal{M}$  is differentiable then the collection of  $\Omega$ -spectra in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  is closed under  $\mathbb{N}$ -indexed homotopy colimits.

EXAMPLE 2.4.3. Recall that a combinatorial model category  $\mathcal{M}$  is called **finitely combinatorial** if the underlying category of  $\mathcal{M}$  is compactly generated and there exist sets of generating cofibrations and trivial cofibrations whose domains and codomains are compact (see [RR15]). The classes of fibrations and trivial fibrations, and hence the class of weak equivalences, are then closed under filtered colimits. Such a model category  $\mathcal{M}$  is differentiable because filtered colimit diagrams in  $\mathcal{M}$  are already filtered homotopy colimit diagrams, while the functor  $\mathrm{colim} : \mathcal{M}^{\mathbb{N}} \rightarrow \mathcal{M}$  preserves finite limits and fibrations (and hence finite homotopy limits).

LEMMA 2.4.4. *Let  $\mathcal{M}$  be a weakly pointed combinatorial model category and let  $f : X \rightarrow Y$  be a map of pre-spectra such that  $X$  is levelwise cofibrant and  $Y$  is an injective fibrant  $\Omega$ -spectrum at  $m$ , i.e. the square*

$$\begin{array}{ccc} Y_{m,m} & \longrightarrow & Y_{m,m+1} \\ \downarrow & & \downarrow \\ Y_{m+1,m} & \longrightarrow & Y_{m+1,m+1} \end{array} \quad (2.4.1)$$

*is homotopy Cartesian. Then we may factor  $f$  as  $X \xrightarrow{f'} X' \xrightarrow{f''} Y$  such that*

- (1)  *$f'$  is a levelwise cofibration and a stable equivalence and the map  $f'_{n,k} : X_{n,k} \rightarrow X'_{n,k}$  is a weak equivalence for every  $n, k$  except  $(n, k) = (m, m)$ .*
- (2)  *$X'$  is an  $\Omega$ -spectra at  $m$ .*

*Proof.* We first note that we may always factor  $f$  as an injective trivial cofibration  $X \rightarrow X''$  followed by an injective fibration  $X'' \rightarrow Y$ . Replacing  $X$  with  $X''$  we may assume without loss of generality that  $f$  is an injective fibration. Let

$$X_{m,m} \rightarrow P \rightarrow Y_{m,m} \times_{[Y_{m,m+1} \times_{Y_{m+1,m+1}} Y_{m+1,m}]} [X_{m,m+1} \times_{X_{m+1,m+1}} X_{m+1,m}]$$

be a factorization in  $\mathcal{M}$  into a cofibration followed by a trivial fibration. By our assumption on  $Y$  the map  $Y_{m,m} \rightarrow Y_{m,m+1} \times_{Y_{m+1,m+1}} Y_{m+1,m}$  is a trivial fibration and hence the composed map  $P \rightarrow X_{m,m+1} \times_{X_{m+1,m+1}} X_{m+1,m}$  is a trivial fibration as well. Associated to the cofibration  $j : X_{m,m} \rightarrow P$  is now a square of  $(\mathbb{N} \times \mathbb{N})$ -diagrams

$$\begin{array}{ccc} (h_{m,m+1} \amalg_{h_{m+1,m+1}} h_{m+1,m}) \otimes X_{m,m} & \longrightarrow & (h_{m,m+1} \amalg_{h_{m+1,m+1}} h_{m+1,m}) \otimes P \\ \downarrow & & \downarrow \\ h_{m,m} \otimes X_{m,m} & \longrightarrow & h_{m,m} \otimes P \end{array} \quad (2.4.2)$$

The rows of these diagrams are stable equivalences and levelwise cofibrations between levelwise cofibrant objects. It follows that the induced map  $i_m \square j : Q \rightarrow h_{m,m} \otimes P$  from the (homotopy)

pushout to  $h_{m,m} \otimes P$  is a stable equivalence and a levelwise cofibration (see Remark 2.1.9). One can easily check that  $i_m \square j$  is an isomorphism in every degree, except in degree  $(m, m)$  where it is the inclusion  $X_{m,m} \rightarrow P$ . We now define  $X'$  as the pushout

$$\begin{array}{ccc} Q & \longrightarrow & h_{m,m} \otimes P \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

where the left vertical map is the natural map. Since  $Q$  and  $X$  are levelwise cofibrant, the resulting map  $X \rightarrow X'$  is a stable equivalence and an isomorphism in all degrees, except in degree  $(m, m)$  where it is the cofibration  $X_{m,m} \rightarrow P$ . we now see that the map  $X \rightarrow X'$  satisfies properties (1) and (2) above by construction.  $\square$

**COROLLARY 2.4.5.** *Let  $\mathcal{M}$  be a weakly pointed combinatorial model category and let  $f : X \rightarrow Y$  be a map in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  between pre-spectra such that  $X$  is levelwise cofibrant and  $Y$  is an injective fibrant  $\Omega$ -spectrum **below**  $n$ , i.e., it is an  $\Omega$ -spectrum at  $m$  for every  $m < n$ . Then we may factor  $f$  as  $X \xrightarrow{f'} L_n X \xrightarrow{f''} Y$  such that  $f'$  is a levelwise cofibration and a stable equivalence,  $L_n X$  is an  $\Omega$ -spectrum below  $n$  and the induced map  $f'[n] : X[n] \rightarrow L_n X[n]$  (see Remark 2.2.8) is a levelwise weak equivalence of pre-spectra. In particular, if the induced map  $f[n] : X[n] \rightarrow Y[n]$  is already a levelwise weak equivalence then  $f$  is a stable equivalence.*

*Proof.* Apply Lemma 2.4.4 consecutively for  $m = n - 1, \dots, 0$  to construct the factorization  $X \rightarrow L_n X \rightarrow Y$  with the desired properties. Note that if  $f[n] : X[n] \rightarrow Y[n]$  is a levelwise equivalence then the induced map  $L_n X[n] \rightarrow Y[n]$  is a levelwise equivalence and since both  $L_n X$  and  $Y$  are  $\Omega$ -spectra below  $n$  the map  $L_n X \rightarrow Y$  must be a levelwise equivalence. It then follows that  $f : X \rightarrow Y$  is a stable equivalence.  $\square$

**COROLLARY 2.4.6.** *Let  $\mathcal{M}$  be a weakly pointed differentiable combinatorial model category and let  $f : X \rightarrow Y$  be a map in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  such that  $X$  is levelwise cofibrant pre-spectrum and  $Y$  is an injective fibrant  $\Omega$ -spectrum. Then there exists a sequence of levelwise cofibrations and stable equivalences*

$$X \rightarrow L_1 X \rightarrow L_2 X \rightarrow \dots$$

*over  $Y$  such that for each  $n$  the map  $X[n] \rightarrow L_n X[n]$  is a levelwise weak equivalence and  $L_n X$  is an  $\Omega$ -spectrum below  $n$ . Furthermore, the induced map  $X \rightarrow L_\infty X \stackrel{\text{def}}{=} \text{colim } L_n X$  is a stable equivalence and  $L_\infty X$  is an  $\Omega$ -spectrum.*

*Proof.* Define the objects  $L_n X$  inductively by requiring  $L_n X \rightarrow L_{n+1} X$  to be the map from  $L_n X$  to an  $\Omega$ -spectrum below  $n + 1$  constructed in Corollary 2.4.5. The resulting sequence is easily seen to have all the mentioned properties.

Since all the maps  $L_n X \rightarrow L_{n+1} X$  are levelwise cofibrations between levelwise cofibrant objects it follows that the map  $X \rightarrow L_\infty X$  is the homotopy colimit in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  of the maps  $X \rightarrow L_n X$ . Since the collection of stable equivalences between pre-spectra is closed under homotopy colimits we may conclude that the map  $X \rightarrow L_\infty X$  is a stable equivalence between pre-spectra. The assumption that  $\mathcal{M}$  is differentiable implies that for each  $m$  the collection of  $\Omega$ -spectra at  $m$  is closed under sequential homotopy colimits. We may therefore conclude that  $L_\infty X$  is an  $\Omega$ -spectrum at  $m$  for every  $m$ , i.e., an  $\Omega$ -spectrum.  $\square$

*Remark 2.4.7.* Since the map  $X_{n,n} \rightarrow (L_n X)_{n,n}$  is a weak equivalence in  $\mathcal{M}$  and  $L_n X$  is a pre-spectrum and an  $\Omega$ -spectrum below  $n$  it follows that the space  $(L_n X)_{0,0}$  is a model for  $n$ -



fold loop object  $\Omega^n X_{n,n}$  in  $\mathcal{M}$ . The above result then asserts that for any pre-spectrum  $X$ , its  $\Omega$ -spectrum replacement  $L_\infty X$  is given in degree  $(k, k)$  by  $\text{hocolim}_n \Omega^n X_{k+n, k+n}$ . In particular  $\mathbb{R}\Omega^\infty X \simeq \text{hocolim}_n \Omega^n X_{n,n}$ .

**COROLLARY 2.4.8.** *Let  $\mathcal{R} : \mathcal{M} \rightarrow \mathcal{N}$  be a differentiable right Quillen functor between weakly pointed combinatorial model categories. Then the right derived Quillen functor  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}} : \mathcal{M}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathcal{N}_{\text{proj}}^{\mathbb{N} \times \mathbb{N}}$  preserves stable equivalences between pre-spectra. If in addition  $\mathbb{R}\mathcal{R}$  detects weak equivalences then  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}$  detects stable equivalences between pre-spectra.*

*Proof.* Let  $f : X \rightarrow Y$  be a stable equivalence between pre-spectra. We may assume without loss of generality that  $X$  is levelwise cofibrant. Let

$$Y \rightarrow L_1 Y \rightarrow L_2 Y \rightarrow \dots$$

be constructed as in Corollary 2.4.6 with respect to the map  $Y \rightarrow *$  and let  $Y_\infty = \text{colim}_n L_n Y$ . Similarly, let

$$X \rightarrow L_1 X \rightarrow L_2 X \rightarrow \dots$$

be a sequence as in Corollary 2.4.6 constructed with respect to the map  $X \rightarrow Y_\infty$ , and let  $X_\infty = \text{colim}_n L_n X$ . Since  $L_n X$  is an  $\Omega$ -spectrum below  $n$  it follows that  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(L_n X)$  is an  $\Omega$ -spectrum below  $n$ . Furthermore, since the map  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(X)[n] \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(L_n X)[n]$  is a levelwise equivalence it follows from the final part of Corollary 2.4.5 that the map  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(X) \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(L_n X)$  is a stable equivalence. By the same argument the map  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(Y) \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(L_n Y)$  is a stable equivalence. Since the maps  $L_n X \rightarrow L_{n+1} X$  are levelwise cofibrations between levelwise cofibrant objects it follows that  $X_\infty \simeq \text{hocolim}_n L_n X$  and  $Y_\infty \simeq \text{hocolim}_n L_n Y$ . Since  $\mathbb{R}\mathcal{R}$  preserves sequential homotopy colimits by assumption we may conclude that the maps  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(X) \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(X_\infty)$  and  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(Y) \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(Y_\infty)$  are stable equivalences. Now since  $X_\infty \rightarrow Y_\infty$  is a stable equivalence between  $\Omega$ -spectra it is also a levelwise weak equivalence. We thus conclude that the map  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(X_\infty) \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(Y_\infty)$  is a levelwise equivalence. The map  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(X) \rightarrow \mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}(Y)$  is hence a stable equivalence in  $\mathcal{N}^{\mathbb{N} \times \mathbb{N}}$  by the 2-out-of-3 property.  $\square$

**COROLLARY 2.4.9.** *Let  $\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$  be a differentiable Quillen pair of weakly pointed left proper combinatorial model categories and let  $n \geq 0$  be a natural number.*

- (1) *If the derived unit  $u_x : X \rightarrow \mathbb{R}\mathcal{R}(\mathcal{L}X)$  either has the property that  $\Omega^n u_x$  is an equivalence for every cofibrant  $X$  or  $\Sigma^n u_x$  is an equivalence for every cofibrant  $X$ , then the derived unit of  $\text{Sp}(\mathcal{L}) \dashv \text{Sp}(\mathcal{R})$  is weak equivalence for every levelwise cofibrant pre-spectrum.*
- (2) *If the derived counit  $\nu_x : \mathbb{L}\mathcal{L}(\mathcal{R}Y) \rightarrow Y$  either has the property that  $\Omega^n \nu_x$  is an equivalence for every fibrant  $Y$  or  $\Sigma^n \nu_y$  is an equivalence for every fibrant  $Y$ , then the derived counit of  $\text{Sp}(\mathcal{L}) \dashv \text{Sp}(\mathcal{R})$  is weak equivalence for every levelwise fibrant pre-spectrum.*

*Proof.* We will only prove the first claim; the second claim follows from a similar argument. Let  $A \in \mathcal{M}^{\mathbb{N} \times \mathbb{N}}$  be a levelwise cofibrant pre-spectrum object in  $\mathcal{M}$ . Since  $\mathcal{R}$  is differentiable we have by Corollary 2.4.8 that  $\mathbb{R}\mathcal{R}^{\mathbb{N} \times \mathbb{N}}$  preserves stable equivalences between pre-spectra. It follows that the derived unit map is equivalent to the map  $u_A : A \rightarrow \mathcal{R}(\mathcal{L}(A)^{\text{proj}})$  which is given levelwise by the derived unit map of the adjunction  $\mathcal{L} \dashv \mathcal{R}$ . If this unit map is levelwise given by a map which becomes an equivalence upon applying  $\Sigma^n$ , then the entire unit map becomes a levelwise equivalence after suspending  $n$  times (recall that suspension in  $\text{Sp}(\mathcal{M})$ , like all homotopy colimits, can be computed levelwise). Since  $\text{Sp}(\mathcal{M})$  is stable this means that  $u_A$  is itself an equivalence.

Now assume that  $u_A$  is given levelwise by a map which becomes an equivalence after applying  $\Omega^n$ . Since  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen adjunction between weakly pointed model categories, the above

map is a map of pre-spectra. It therefore suffices to check that the induced map

$$A^{\text{fib}} \longrightarrow \mathcal{R}(\mathcal{L}(A)^{\text{proj}})^{\text{fib}}$$

on the (explicit) fibrant replacements provided by Corollary 2.4.6 is a levelwise equivalence. By Remark 2.4.7 this map is given at level  $(k, k)$  by the induced map

$$\text{hocolim}_i \Omega^i A_{i+k, i+k} \longrightarrow \text{hocolim}_i \Omega^i \mathbb{R}\mathcal{R}(\mathcal{L}(A_{k+i, k+i})).$$

This map is induced from a map of sequences which is a weak equivalence for all  $i \geq n$  by our assumption, and so the desired result follows.  $\square$

### 3. Tangent categories and tangent bundles

In §2 we have set up a convenient formalism for stabilizing model categories. In this section we will use this formalism in order to define and study the **tangent model categories** of a given model category.

DEFINITION 3.0.1. Let  $\mathcal{M}$  be a left proper combinatorial model category and  $A \in \mathcal{M}$  an object. We will denote by

$$\mathcal{T}_A \mathcal{M} \stackrel{\text{def}}{=} \text{Sp}(\mathcal{M}_{A//A})$$

the stabilization of  $\mathcal{M}_{A//A}$ , and refer to it as the **tangent category to  $\mathcal{M}$  at  $A$** . Here  $\mathcal{M}_{A//A} := (\mathcal{M}_{/A})_{\text{id}_A/} = (\mathcal{M}_{/A})_*$  is the (combinatorial, left proper) model category of over-under  $A$ -objects in  $\mathcal{M}$ , with its induced model structure.

EXAMPLE 3.0.2. When  $\mathcal{M}$  is the category of simplicial sets endowed with the Kan-Quillen model structure and  $X \in \mathcal{M}$  is a simplicial set then the tangent category  $\mathcal{T}_X \mathcal{M}$  is a model for the theory of **parametrized spectra** over  $X$  (see [MS06], [ABG11, §B] and the discussion in [HNP16, §2.3]).

EXAMPLE 3.0.3. When  $\mathcal{M}$  is a sufficiently nice model for commutative ring spectra (such as the one constructed in [Shi04]), and  $R$  is a given commutative ring spectrum, then the tangent model category  $\mathcal{T}_R \mathcal{M}$  is Quillen equivalent to the model category of  $R$ -modules, see [BM05] (which works in the topological setting of  $S$ -algebras and the Bousfield-Friedlander stabilization).

By Proposition 2.2.5 we may consider  $\mathcal{T}_A \mathcal{M}$  as a model for the **tangent  $\infty$ -category** of  $\mathcal{M}_\infty$  at the object  $A$ , in the sense of [Lur14, §7.3.1]. Our goal in this section is to use the formalism of [HP15] in order to assemble the various tangent categories  $\mathcal{T}_A \mathcal{M}$  into a global category  $\mathcal{T}\mathcal{M} = \int_{\mathcal{M}} \text{Sp}(\mathcal{M}_{A//A})$  with a suitable model structure, yielding a model for the **tangent bundle  $\infty$ -category**  $\mathcal{T}\mathcal{M}_\infty$ . We will begin in §3.1 with recalling the formalism of [HP15] and establishing some preliminary results. These will be used in §3.2 and §3.3 to construct the tangent bundle  $\mathcal{T}\mathcal{M}$  as a model category and establish its basic properties. We will then explain in §3.4 how this global point of view can be used to describe the tangent categories of functor categories.

#### 3.1 Preliminaries

Recall that a suitable variant of the classical Grothendieck correspondence asserts that if  $\mathcal{C}$  is a category, then the data of a (pseudo-)functor from  $\mathcal{C}$  to the  $(2, 1)$ -category of categories and adjunctions is equivalent to the data of a functor  $\mathcal{D} \longrightarrow \mathcal{C}$  which is simultaneously a Cartesian and a coCartesian fibration. In this paper we will make use of the model categorical analogue of this result, as developed in [HP15]. Let  $\text{ModCat}$  be the  $(2, 1)$ -category of model categories and Quillen adjunctions and let  $\mathcal{F} : \mathcal{M} \longrightarrow \text{ModCat}$  be a functor whose domain  $\mathcal{M}$  carries a

model structure. Given a map  $f : A \rightarrow B$  in  $\mathcal{M}$ , we will denote by  $f_! : \mathcal{F}(A) \xrightarrow{\perp} \mathcal{F}(B) : f^*$  the Quillen adjunction associated to  $f$  by  $\mathcal{F}$ . We will say that  $\mathcal{F}$  is **relative** if  $f_! \dashv f^*$  is a Quillen equivalence whenever  $f$  is a weak equivalence. We will say that  $\mathcal{F}$  is **left proper** if  $f_!$  preserves weak equivalences whenever  $f$  is a trivial cofibrations. Dually, we will say that  $\mathcal{F}$  is **right proper** if  $f^*$  preserves weak equivalences whenever  $f$  is a trivial fibration. Finally, we will say that  $\mathcal{F}$  is **proper** if it is both left proper and right proper.

Theorem 3.0.12 of [HP15] asserts that the notion of a proper relative functor  $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$  is equivalent to a suitable notion of a **model fibration**  $\pi : \mathcal{N} \rightarrow \mathcal{M}$ . In particular, the underlying category of  $\mathcal{N}$  is the Cartesian-coCartesian fibration associated to the underlying functor from  $\mathcal{M}$  to categories and adjunctions. We will refer to the model structure on  $\mathcal{N} \cong \int_{\mathcal{M}} \mathcal{F}$  determined by this correspondence as the **integral model structure**. This model structure enjoys many favorable formal properties. For example, the projection  $\pi$  is both a left and a right Quillen functor and the integral model structure is compatible with base change.

In this section we will establish some general results showing that the class of proper relative functors (or, equivalently, the class of model fibrations) is closed under various operations, which we will need in order to construct tangent bundles.

**LEMMA 3.1.1.** *Let  $\mathcal{M}$  be a model category and  $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$  be a proper relative functor such that each  $\mathcal{F}(A)$  is a combinatorial model category for every  $A \in \mathcal{M}$ . Then for any small category  $\mathcal{J}$ , the functors  $\mathcal{F}_{\text{inj}}^{\mathcal{J}}(A) = (\mathcal{F}(A)^{\mathcal{J}})_{\text{inj}}$  and  $\mathcal{F}_{\text{proj}}^{\mathcal{J}}(A) = (\mathcal{F}(A)^{\mathcal{J}})_{\text{proj}}$  are proper and relative. If  $\mathcal{J}$  is a Reedy category then the functor  $\mathcal{F}_{\text{Reedy}}^{\mathcal{J}}(A) = (\mathcal{F}(A)^{\mathcal{J}})_{\text{Reedy}}$  is proper and relative as well.*

*Proof.* This follows directly from the fact that Quillen equivalences induces Quillen equivalences on functor categories and that weak equivalences in functor categories are levelwise.  $\square$

**LEMMA 3.1.2.** *Let  $\mathcal{L} : \mathcal{M} \xrightarrow{\perp} \mathcal{N} : \mathcal{R}$  be a Quillen adjunction and let  $LM, LN$  be left Bousfield localizations of  $\mathcal{M}$  and  $\mathcal{N}$  respectively. Assume that  $\mathbb{R}\mathcal{R}$  preserves local objects. Then  $\mathcal{L} \xrightarrow{\perp} \mathcal{R}$  descends uniquely to a Quillen adjunction  $\overline{\mathcal{L}} : LM \xrightarrow{\perp} LN : \overline{\mathcal{R}}$ . Furthermore*

- (i) *If  $\mathbb{R}\mathcal{R}$  preserves and detects local objects and  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen equivalence then  $\overline{\mathcal{L}} \dashv \overline{\mathcal{R}}$  is a Quillen equivalence.*
- (ii) *If  $\mathcal{L}$  preserves weak equivalences then  $\overline{\mathcal{L}}$  preserves weak equivalence. If  $\mathcal{R}$  preserves weak equivalences and  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen equivalence then  $\overline{\mathcal{R}}$  preserves weak equivalences.*

*Proof.* The fact that  $\mathcal{L} \xrightarrow{\perp} \mathcal{R}$  descends to  $\overline{\mathcal{L}} \xrightarrow{\perp} \overline{\mathcal{R}}$  follows from [Hir03, Theorem 3.1.6]. Claim (1) is now a combination of Proposition [Hir03, Proposition 3.1.12] and Lemma [Hir03, Theorem 3.3.20]. Let us prove Claim (2). Since cofibrant replacements in  $\mathcal{M}$  and  $LM$  are the same it follows that if  $\mathcal{L}$  preserves weak equivalences then  $\overline{\mathcal{L}} \simeq \mathbb{L}\overline{\mathcal{L}}$  and hence  $\overline{\mathcal{L}}$  preserves weak equivalences. Now suppose that  $\mathcal{R}$  preserves weak equivalences and that  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen equivalence. By (1) we know  $\overline{\mathcal{L}} \dashv \overline{\mathcal{R}}$  is a Quillen equivalence and hence  $\mathbb{L}\overline{\mathcal{L}} \simeq \mathbb{L}\mathcal{L}$  detects local equivalences. It will hence suffice to show that if  $X, Y$  are fibrant in  $\mathcal{N}$  and  $f : X \rightarrow Y$  is a local weak equivalence then the induced map  $\mathcal{L}(\mathcal{R}(X)^{\text{cof}}) \rightarrow \mathcal{L}(\mathcal{R}(Y)^{\text{cof}})$  is a local weak equivalence. But this now follows from the fact that the derived counit map of  $\mathcal{L} \dashv \mathcal{R}$  is a weak equivalence.  $\square$

**COROLLARY 3.1.3 (Localization in families).** *Let  $\mathcal{M}$  be a model category and  $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$  be a proper relative functor. Let  $\mathcal{F}^{\text{loc}} : \mathcal{M} \rightarrow \text{ModCat}$  be a functor together with a natural transformation  $F \Rightarrow \mathcal{F}^{\text{loc}}$  which is a component-wise left Bousfield localization (of model categories). Suppose that  $\mathcal{F}^{\text{loc}}$  satisfies the following condition: for every weak equivalence  $f : A \xrightarrow{\sim} B \in \mathcal{M}$ ,  $\mathbb{R}f^*$  preserves and detects local objects. Then  $\mathcal{F}^{\text{loc}}$  is proper and relative.*

*Proof.* Let  $f : A \rightarrow B$  be a weak equivalence. By Lemma 3.1.2(1) the Quillen equivalence  $f_! \dashv f^*$  descends to a Quillen equivalence  $f_!^{\text{loc}} : \mathcal{F}^{\text{loc}}(A) \xrightleftharpoons{\perp} \mathcal{F}^{\text{loc}}(B) : (f^*)^{\text{loc}}$ , and so  $\mathcal{F}^{\text{loc}}$  is relative. To finish the proof we note that if  $f$  is a trivial cofibration then  $f_!^{\text{loc}}$  preserves weak equivalences by Lemma 3.1.2(2) and if  $f$  is a trivial fibration then  $f_! \xrightleftharpoons{\perp} f^*$  is a Quillen equivalence and hence  $(f^*)^{\text{loc}}$  preserves weak equivalences by Lemma 3.1.2(2). It follows that  $\mathcal{F}^{\text{loc}}$  is proper as well.  $\square$

We shall now consider the operation of forming slice and coslice categories.

**CONSTRUCTION 3.1.4.** Let  $\mathcal{L} : \mathcal{C} \xrightleftharpoons{\perp} \mathcal{D} : \mathcal{R}$  be an adjunction between categories which admit finite limits and colimits. Consider a map in  $\mathcal{D}$  of the form  $f : \mathcal{L}(A) \rightarrow B$  and let  $f^{\text{ad}} : A \rightarrow \mathcal{R}(B)$  be its adjoint. Then one obtains an induced adjunction

$$\mathcal{L}_{/f} : \mathcal{C}_{/A} \xrightleftharpoons{\perp} \mathcal{C}_{/B} : \mathcal{R}_{/f},$$

where  $\mathcal{L}_{/f}(X \rightarrow A)$  is the composed map  $\mathcal{L}(X) \rightarrow \mathcal{L}(A) \xrightarrow{f} B$  and  $\mathcal{R}_{/f}(Y \rightarrow B)$  is the projection  $\mathcal{R}(Y) \times_{\mathcal{R}(B)} A \rightarrow A$  (where the pullback is done along the map  $f^{\text{ad}} : A \rightarrow \mathcal{R}(B)$ ). Similarly, one obtains an induced adjunction

$$\mathcal{L}_{f/} : \mathcal{C}_{A/} \xrightleftharpoons{\perp} \mathcal{C}_{B/} : \mathcal{R}_{f/},$$

where  $\mathcal{L}_{f/}(A \rightarrow X) = B \rightarrow \mathcal{L}(X) \amalg_{\mathcal{L}(A)} B$  and  $\mathcal{R}_{f/}(B \rightarrow Y)$  is the composed map  $A \xrightarrow{f^{\text{ad}}} \mathcal{R}(B) \rightarrow \mathcal{R}(Y)$ . It is not hard to check that if  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen adjunction between model categories then  $\mathcal{L}_{/f} \dashv \mathcal{R}_{/f}$  and  $\mathcal{L}_{f/} \dashv \mathcal{R}_{f/}$  are Quillen adjunctions as well with respect to the corresponding slice/coslice model structures.

*Remark 3.1.5.* In construction 3.1.4, if  $A$  is initial or  $B$  is terminal then the choice of  $f$  is unique. In particular, any adjunction  $\mathcal{L} \dashv \mathcal{R}$  induces canonical adjunctions  $\mathcal{L}_* : \mathcal{C}_* \xrightleftharpoons{\perp} \mathcal{D}_* : \mathcal{R}_*$  and  $\mathcal{L}_{\text{aug}} : \mathcal{C}_{\text{aug}} \xrightleftharpoons{\perp} \mathcal{D}_{\text{aug}} : \mathcal{R}_{\text{aug}}$ , where for a category  $\mathcal{E}$  we denote by  $\mathcal{E}_* = \mathcal{E}_{*/}$  the category of pointed objects (i.e., objects under the terminal object) and by  $\mathcal{E}_{\text{aug}} = \mathcal{E}_{/\emptyset}$  the category of augmented objects (i.e., objects over the initial object).

*Remark 3.1.6.* In the setting of construction 3.1.4 we have a natural isomorphism  $\mathcal{U}_{/B} \circ \mathcal{L}_{/f} \cong \mathcal{L} \circ \mathcal{U}_{/A}$  of left Quillen functors where  $\mathcal{U}_{/A} : \mathcal{M}_{/A} \rightarrow \mathcal{M}$  and  $\mathcal{U}_{/B} : \mathcal{N}_{/B} \rightarrow \mathcal{N}$  are the corresponding forgetful functors. If  $f^{\text{ad}} : A \rightarrow \mathcal{R}(B)$  is an isomorphism then we also have an isomorphism  $\mathcal{U}_{/A} \circ \mathcal{R}_{/f} \cong \mathcal{R} \circ \mathcal{U}_{/B}$ . Since  $\mathcal{U}_{/A}$  and  $\mathcal{U}_{/B}$  detect weak equivalences it follows that if  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen equivalence and  $f$  is an isomorphism then  $\mathcal{L}_{/f} \dashv \mathcal{R}_{/f}$  is a Quillen equivalence. Similarly, if  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen equivalence and  $f : \mathcal{L}(A) \rightarrow B$  is an isomorphism then  $\mathcal{L}_{f/} \dashv \mathcal{R}_{f/}$  is a Quillen equivalence.

Under suitable hypothesis the content of Remark 3.1.6 can be strengthened:

**LEMMA 3.1.7.** *Let  $\mathcal{L} : \mathcal{M} \xrightleftharpoons{\perp} \mathcal{N} : \mathcal{R}$  be a Quillen adjunction between left proper (resp. right proper) model categories and let  $f : \mathcal{L}(A) \rightarrow B$  be a map in  $\mathcal{N}$ . Then:*

- (1) *If the composed map  $f' : \mathcal{L}(A^{\text{cof}}) \rightarrow \mathcal{L}(A) \rightarrow B$  be a weak equivalence and  $\mathcal{L} \dashv \mathcal{R}$  is a Quillen equivalence then  $\mathcal{L}_{f/} \dashv \mathcal{R}_{f/}$  (resp.  $\mathcal{L}_{/f} \dashv \mathcal{R}_{/f}$ ) is a Quillen equivalence.*
- (2) *If  $\mathcal{R}$  (resp.  $\mathcal{L}$ ) preserves weak equivalences then  $\mathcal{R}_{f/}$  (resp.  $\mathcal{L}_{/f}$ ) preserves weak equivalences. If  $\mathcal{L}$  (resp.  $\mathcal{R}$ ) preserves weak equivalences and  $f : \mathcal{L}(A) \rightarrow B$  (resp.  $f^{\text{ad}} : A \rightarrow \mathcal{R}(B)$ ) is a trivial cofibration (resp. trivial fibration) then  $\mathcal{L}_{f/}$  preserves weak equivalences.*

*Proof.* Let us prove the case where  $\mathcal{M}$  and  $\mathcal{N}$  are left proper. The right proper case can be deduced by applying the same argument to the opposite adjunction  $\mathcal{M}^{\text{op}} \rightleftarrows \mathcal{N}^{\text{op}}$ . To prove (1), we first note that the composed  $\mathcal{M}_{A^{\text{cof}}/} \rightleftarrows \mathcal{M}_{A/} \rightleftarrows \mathcal{N}_{B/}$  is naturally isomorphic to the adjunction associated by Construction 3.1.4 to  $f' : \mathcal{L}(A^{\text{cof}}) \rightarrow B$ . Since  $\mathcal{M}$  is left proper the adjunction  $\mathcal{M}_{A^{\text{cof}}/} \rightleftarrows \mathcal{M}_{A/}$  is a Quillen equivalence. It will hence suffice to prove the claim for  $f'$ . We now note that  $\mathcal{L}_{f'/} : \mathcal{M}_{A^{\text{cof}}/} \rightleftarrows \mathcal{N}_{A/}$  is naturally isomorphic to the composed adjunction  $\mathcal{M}_{A^{\text{cof}}/} \rightleftarrows \mathcal{N}_{\mathcal{L}(A^{\text{cof}})/} \rightleftarrows \mathcal{N}_{B/}$ . By our assumption  $f'$  is a weak equivalence and since  $\mathcal{N}$  is left proper the adjunction  $\mathcal{N}_{\mathcal{L}(A^{\text{cof}})/} \rightleftarrows \mathcal{N}_{B/}$  is a Quillen equivalence. It will hence suffice to show that the adjunction  $\mathcal{M}_{A^{\text{cof}}/} \rightleftarrows \mathcal{M}_{\mathcal{L}(A^{\text{cof}})/}$  is a Quillen adjunction. In other words, we have reduced to the case where  $A$  is cofibrant and  $f$  is the identity. But this now follows from Remark 3.1.6.

Let us now prove (2). First it is clear that if  $\mathcal{R}$  preserves weak equivalence then  $\mathcal{R}_{f/}$  which is defined by  $\mathcal{R}_{f/}(\mathcal{L}(A) \rightarrow Y) = A \rightarrow \mathcal{R}(\mathcal{L}(A)) \rightarrow \mathcal{R}(Y)$  preserves weak equivalences. Now assume that  $\mathcal{L}$  preserves weak equivalences and that  $f : \mathcal{L}(A) \rightarrow B$  is a trivial cofibration. Then we can write  $\mathcal{L}_{f/}$  as a composition  $\mathcal{M}_{A/} \rightarrow \mathcal{M}_{\mathcal{L}(A)/} \rightarrow \mathcal{M}_{B/}$  where the first functor clearly preserve weak equivalences and the second preserves weak equivalences because it is given by a pushout along a trivial cofibration.  $\square$

DEFINITION 3.1.8. Let  $\mathcal{M}$  be a model category and  $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$  be a proper relative functor. Let  $\pi : \int_{\mathcal{F}} \mathcal{M} \rightarrow \mathcal{M}$  be the model fibration associated to  $\mathcal{F}$  and let  $s : \mathcal{M} \rightarrow \int_{\mathcal{M}} \mathcal{F}$  be a section of  $\pi$  (not necessarily left or right Quillen). We will denote by  $\mathcal{F}_{s/} : \mathcal{M} \rightarrow \text{ModCat}$  the functor which is given on the level of objects by

$$\mathcal{F}_{s/}(A) = \mathcal{F}(A)_{s(A)/}$$

and on the level of morphisms by the adjunctions

$$\mathcal{F}_{s/}(f : A \rightarrow B) = (f!)_{s(f)/} : \mathcal{F}(A)_{s(A)/} \rightleftarrows \mathcal{F}(B)_{s(B)/} : (f^*)_{s(f)/}.$$

of Construction 3.1.4. Similarly, we define  $\mathcal{F}_{/s} : \mathcal{M} \rightarrow \text{ModCat}$  using the associated slice categories.

COROLLARY 3.1.9 (Slicing in families). *Let  $\mathcal{M}$  be a model category and  $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$  be a proper relative functor such that  $\mathcal{F}(A)$  is left proper (resp. right proper) for every  $A \in \mathcal{M}$ . Let  $\pi : \int_{\mathcal{F}} \mathcal{M} \rightarrow \mathcal{M}$  be the model fibration associated to  $\mathcal{F}$  and let  $s : \mathcal{M} \rightarrow \int_{\mathcal{M}} \mathcal{F}$  be a section of  $\pi$  (not necessarily left or right Quillen) such that  $s$  preserves weak equivalences and trivial cofibrations (resp. trivial fibrations). Then the functor  $\mathcal{F}_{s/}$  (resp.  $\mathcal{F}_{/s}$ ) of Definition 3.1.8 is proper and relative.*

*Proof.* This follows directly from Lemma 3.1.7.  $\square$

COROLLARY 3.1.10. *Let  $\mathcal{M}$  be a model category and  $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$  be a proper relative functor satisfying the following conditions:*

- (1)  $\mathcal{F}(A)$  is a left proper combinatorial model category for every  $A \in \mathcal{M}$ .
- (2) If  $f : A \rightarrow B$  is a trivial cofibration in  $\mathcal{M}$  then the terminal map  $f_!(*) \rightarrow *$  is a trivial cofibration.

*Then the functor  $\mathcal{F}_*(A) = \mathcal{F}(A)_*$  is proper and relative.*

*Proof.* Let  $s : \mathcal{M} \rightarrow \int_{\mathcal{M}} \mathcal{F}$  be the terminal section. Then  $s$  sends trivial cofibrations to trivial cofibrations by Condition (2) above. Furthermore, if  $f : A \rightarrow B$  is a weak equivalence then  $f_!$  is a left Quillen equivalence and hence the composed map  $f_!(\ast^{\text{cof}}) \rightarrow f_!(\ast) \rightarrow \ast$  is a weak

equivalence in  $\mathcal{F}(B)$ . It follows that  $s(f)$  is a weak equivalence in  $\int_{\mathcal{M}} \mathcal{F}$  and so the desired result is now a particular case of Corollary 3.1.9.  $\square$

### 3.2 Tangent bundles

Our goal in this section is to use the formalism of [HP15] in order to assemble the various tangent categories  $\mathcal{T}_A \mathcal{M}$  into a global category  $\mathcal{T}\mathcal{M} = \int_{\mathcal{M}} \mathrm{Sp}(\mathcal{M}_{A//A})$  with a suitable model structure, yielding a model for the **tangent bundle**  $\infty$ -category  $\mathcal{T}\mathcal{M}_{\infty}$ . We begin by observing that the constructions developed in the previous section allow us to form the **stabilization** of a proper and relative functor:

**PROPOSITION 3.2.1.** *Let  $\mathcal{M}$  be a model category and  $\mathcal{F} : \mathcal{M} \rightarrow \mathrm{ModCat}$  a proper relative functor satisfying the assumptions of Corollary 3.1.10. Then the functor  $A \mapsto \mathrm{Sp}(\mathcal{F}_*(A))$  is proper and relative.*

*Proof.* Combine Corollary 3.1.10, Lemma 3.1.1 and Corollary 3.1.3 (where the assumptions of Corollary 3.1.3 are satisfied since any levelwise right Quillen equivalence between  $(\mathbb{N} \times \mathbb{N})$ -functor categories preserves and detects  $\Omega$ -spectra).  $\square$

We will apply the above proposition to the family of model categories  $A \mapsto \mathcal{M}_{/A}$  where  $\mathcal{M}$  is a combinatorial proper model category. We first recall the following:

**LEMMA 3.2.2.** *Let  $\mathcal{M}$  be a proper combinatorial model category. Then the functor  $\mathcal{M} \rightarrow \mathrm{ModCat}$  given by  $A \mapsto \mathcal{M}_{/A}$  is proper and relative and satisfies the conditions of Corollary 3.1.10.*

*Proof.* The first part is [HP15, §6.1]. The second part follows from the fact that if  $f : A \rightarrow B$  is a map in  $A$  then the map  $f_!(*) \rightarrow *$  in  $\mathcal{M}_{/B}$  can be identified with  $f$  itself.  $\square$

**COROLLARY 3.2.3.** *Let  $\mathcal{M}$  be a proper combinatorial model category. Then the functor  $\mathcal{M} \rightarrow \mathrm{ModCat}$  given by  $A \mapsto \mathrm{Sp}(\mathcal{M}_{A//A}) = \mathrm{Sp}((\mathcal{M}_{/A})_*)$  is proper and relative.*

*Remark 3.2.4.* Proper combinatorial model categories include the category  $\mathcal{S}$  of simplicial sets, as well as every simplicial presheaf category, and every left exact localization thereof. On the algebraic side, a theorem of Rezk ([Rez02, Theorem A]) shows that the model category of algebras over any cofibrant simplicial multi-sorted algebraic theory (and in particular, the theory of simplicial algebras over a cofibrant simplicial operad) is proper. On a similar note, if  $\mathcal{M}$  is a proper model category in which every object is cofibrant and  $\mathcal{P}$  is a cofibrant single-colored operad in  $\mathcal{M}$  then the work of Fresse ([Fre09, §17.4]) shows that the model category of  $\mathcal{P}$ -algebras in  $\mathcal{M}$  is proper (see Remark 4.2.2).

**DEFINITION 3.2.5.** Let  $\mathcal{M}$  be a proper combinatorial model category. We will refer to the model category arising from Corollary 3.2.3 as the **tangent model category** of  $\mathcal{M}$  and denote it by

$$\mathcal{T}\mathcal{M} := \int_{A \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{A//A})$$

We will refer to the model fibration  $\pi : \mathcal{T}\mathcal{M} \rightarrow \mathcal{M}$  as the **tangent model fibration** of  $\mathcal{M}$ .

Recall that for every  $A \in \mathcal{M}$  we have a canonical adjunction  $\Sigma_+^{\infty} : \mathcal{M}_{/A} \rightleftarrows \mathrm{Sp}(\mathcal{M}_{/A}) : \Omega_+^{\infty}$  (see §2.2). The collection of right Quillen functors  $\Omega_+^{\infty} : \mathrm{Sp}(\mathcal{M}_{A//A}) \rightarrow \mathcal{M}_{/A}$  yields a right Quillen morphism  $\mathrm{Sp}(\mathcal{M}_{(-)//(-)}) \Rightarrow \mathcal{M}_{/(-)}$  in the sense of [HP15, §4] and hence an induced right Quillen functor  $\mathcal{T}\mathcal{M} \rightarrow \int_{A \in \mathcal{M}} \mathcal{M}_{/A} = \mathcal{M}^{[1]}$  over  $\mathcal{M}$ .

PROPOSITION 3.2.6. *The diagram*

$$\begin{array}{ccc}
 (\mathcal{T}\mathcal{M})_\infty & \xrightarrow{\quad} & (\mathcal{M}_\infty)^{[1]} \\
 \pi_\infty \searrow & & \swarrow \text{ev}_1 \\
 & \mathcal{M}_\infty &
 \end{array} \tag{3.2.1}$$

*exhibits  $(\mathcal{T}\mathcal{M})_\infty$  as the tangent bundle to  $\mathcal{M}_\infty$  in the sense of [Lur14, Definition 7.3.1.9].*

*Proof.* By [HP15, Proposition 3.1.2] the map  $\pi_\infty$  is a coCartesian map whose fibers are the underlying  $\infty$ -categories of  $\text{Sp}(\mathcal{M}_{A//A})$ . By [Lur09, Proposition A.3.7.6] these fibers are presentable and by Corollary 2.2.6 they are also stable. Proposition 2.2.5 now implies that the map  $(\int_{\mathcal{M}} \text{Sp}(\mathcal{M}_{A//A}))_\infty \rightarrow (\mathcal{M}_\infty)^{[1]}$  is a stable envelope of  $(\mathcal{M}_\infty)^{[1]} \rightarrow \mathcal{M}_\infty$  in the sense of [Lur14, Definition 7.3.1.1] and hence exhibits  $(\int_{\mathcal{M}} \text{Sp}(\mathcal{M}_{A//A}))_\infty$  as the tangent bundle of  $\mathcal{M}_\infty$ .  $\square$

PROPOSITION 3.2.7. *A Quillen pair  $\mathcal{L} : \mathcal{M} \xrightleftharpoons[\perp]{} \mathcal{N} : \mathcal{R}$  of proper combinatorial model categories induces a Quillen pair  $\mathcal{T}\mathcal{L} : \mathcal{T}\mathcal{M} \xrightleftharpoons[\perp]{} \mathcal{T}\mathcal{N} : \mathcal{T}\mathcal{R}$  such that the following diagram of Quillen functors commutes:*

$$\begin{array}{ccc}
 \mathcal{T}\mathcal{M} & \xrightleftharpoons[\perp]{} & \mathcal{T}\mathcal{N} \\
 \pi_{\mathcal{M}} \downarrow & & \downarrow \pi_{\mathcal{N}} \\
 \mathcal{M} & \xrightleftharpoons[\perp]{} & \mathcal{N}
 \end{array}$$

*Proof.* For any object  $A \in \mathcal{M}$ , the Quillen adjunction  $\mathcal{L} \dashv \mathcal{R}$  induces a Quillen adjunction

$$\text{Sp}(\mathcal{M}_{A//A}) \xrightleftharpoons[\perp]{} \text{Sp}(\mathcal{N}_{\mathcal{L}A//\mathcal{L}A})$$

which fits into a left Quillen morphism  $\text{Sp}(\mathcal{M}_{(-)//(-)}) \Rightarrow \text{Sp}(\mathcal{N}_{(-)//(-)})$  in the sense of [HP15, §4]. We then get a Quillen adjunction  $\mathcal{T}\mathcal{L} : \mathcal{T}\mathcal{M} \xrightleftharpoons[\perp]{} \mathcal{T}\mathcal{N} : \mathcal{T}\mathcal{R}$  which is compatible with the corresponding tangent model fibrations (see [HP15, Theorem 4.1.3]).  $\square$

### 3.3 Tensor structures on the tangent bundle

Let  $\mathcal{M}$  be a proper combinatorial model category which is tensored over a combinatorial symmetric monoidal (or SM for short) model category  $\mathbf{S}$ . Our goal in this section is to show that in favorable cases the tangent bundle  $\mathcal{T}\mathcal{M}$  inherits a natural tensor structure over  $\mathbf{S}$ . Such a structure is useful when considering the tangent bundles of (enriched) functor categories, as we will see in §3.4.

Recall that if  $A$  is a set then  $A_* = A \cup \{*\}$  is the free pointed set generated from  $A$ . We note that the functor  $A \mapsto A_*$  is a monoidal functor from sets with Cartesian products to pointed sets with smash product. In particular, if  $\mathcal{J}$  is a category, then we may apply the functor  $(-)_*$  to all its hom sets and obtain a category enriched in pointed sets. We may consider such a category as a category in which all hom sets are equipped with a distinguished 0-morphism, such that the composition of a 0-morphism and any other morphism is again a 0-morphism. We will denote by  $\mathcal{J}_*$  the category obtained from  $\mathcal{J}$  by first freely adding an object  $*$  and then pointifying all the hom sets as above. More explicitly, the object set of  $\mathcal{J}_*$  is  $\text{Ob}(\mathcal{J}) \cup \{*\}$ , and we have  $\text{Hom}_{\mathcal{J}_*}(i, j) = \text{Hom}_{\mathcal{J}}(i, j)_*$  for every  $i, j \in \mathcal{J}$ , and  $\text{Hom}_{\mathcal{J}_*}(i, *) = \text{Hom}_{\mathcal{J}_*}(*, i) = \{*\}$  for every  $i \in \mathcal{J}$ . We may consider  $\mathcal{J}_*$  as obtained from  $\mathcal{J}$  by **freely adding a zero object**.

Now suppose that  $\mathcal{J} = (\mathcal{J}, \mathcal{J}^+, \mathcal{J}^-)$  is a **Reedy category** (see [Hov99, Definition 5.2.1]). Then it is easy to see that  $\mathcal{J}_*$  is again a Reedy category, where we consider  $*$   $\in \mathcal{J}_*$  as being the unique object of degree 0 and such that for every  $i$  the unique map  $*$   $\longrightarrow i$  is in  $\mathcal{J}_*^+ = (\mathcal{J}_*)^+$  and the unique map  $i \longrightarrow *$  is in  $\mathcal{J}_*^-$ .

LEMMA 3.3.1. *Let  $\mathcal{J}$  be a Reedy category and  $\mathcal{M}$  a proper model category. Then we have a natural equivalence of categories*

$$\int_{\mathcal{M}} (\mathcal{M}_{A//A})_{\text{Reedy}}^{\mathcal{J}} \simeq \mathcal{M}_{\text{Reedy}}^{\mathcal{J}_*}$$

*identifying the integral model structure on the left with the Reedy model structure on the right.*

*Proof.* One can easily verify that the functor  $\text{ev}_* : \mathcal{M}^{\mathcal{J}_*} \longrightarrow \mathcal{M}$  is a biCartesian fibration which is classified by the functor  $\mathcal{M} \longrightarrow \text{Cat}$  sending  $A$  to  $(\mathcal{M}_{A//A})^{\mathcal{J}}$ . It follows that the two model categories have equivalent underlying categories. It therefore suffices to show that they have the same cofibrations and trivial cofibrations. We will treat the case of cofibrations. The same proof applies as well to trivial cofibrations.

Let  $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$  be a map in  $\mathcal{M}^{\mathcal{J}_*}$  which projects to  $f := \text{ev}_*(\varphi) : \mathcal{F}(*) \longrightarrow \mathcal{G}(*)$  and let  $f_!$  be the induced functor between the fibers of  $\text{ev}_*$ . Under the equivalence of the previous paragraph, the induced map  $f_! \mathcal{F} \longrightarrow \mathcal{G}$  corresponds to a map  $\mathcal{F}' \longrightarrow \mathcal{G}|_{\mathcal{J}}$  of functors from  $\mathcal{J}$  to  $\mathcal{M}_{\mathcal{G}(*)//\mathcal{F}(*)}^{\mathcal{J}}$ , where  $\mathcal{F}'$  is simply given by  $\mathcal{F}'(i) = \mathcal{F}(i) \coprod_{\mathcal{F}(*)} \mathcal{G}(*)$ . Since the forgetful functor  $(\mathcal{M}_{A//A})^{\mathcal{J}} \longrightarrow \mathcal{M}^{\mathcal{J}}$  preserves and detects Reedy (trivial) cofibrations, unwinding the definitions it suffices to show the following: if  $f$  is a cofibration, then  $\varphi$  is a Reedy cofibration if and only if the induced map  $\mathcal{F}' \longrightarrow \mathcal{G}|_{\mathcal{J}}$  is a Reedy cofibration in  $\mathcal{M}^{\mathcal{J}}$ .

For an object  $i \in \mathcal{J}$  let us denote by  $L_i^{\mathcal{J}} : \mathcal{M}^{\mathcal{J}} \longrightarrow \mathcal{M}$  and  $L_i^{\mathcal{J}_*} : \mathcal{M}^{\mathcal{J}_*} \longrightarrow \mathcal{M}$  the corresponding  $i$ 'th latching object functors. Our goal is to show that for  $i \in \mathcal{J}$ , the map

$$L_i^{\mathcal{J}_*}(\mathcal{G}) \coprod_{L_i^{\mathcal{J}_*}(\mathcal{F})} \mathcal{F}(i) \longrightarrow \mathcal{G}(i) \quad (3.3.1)$$

is a cofibration if and only if the map

$$L_i^{\mathcal{J}}(\mathcal{G}|_{\mathcal{J}}) \coprod_{L_i^{\mathcal{J}}(\mathcal{F}')} \mathcal{F}'(i) \longrightarrow \mathcal{G}(i) \quad (3.3.2)$$

is a cofibration. For an object  $i \in \mathcal{J}$  let  $\mathcal{J}_{/i}^+ \subseteq \mathcal{J}_{/i}$  be subcategory whose objects are the non-identity maps  $j \rightarrow i$  in  $\mathcal{J}^+$  and whose morphisms are maps in  $\mathcal{J}^+$  over  $i$ , and let  $\mathcal{J}_{*/i}^+$  be the defined similarly. Note that  $\mathcal{J}_{*/i}^+$  is obtained from  $\mathcal{J}_{/i}^+$  by freely adding an initial object. Consequently, the data of a diagram  $\mathcal{F} : \mathcal{J}_{*/i}^+ \longrightarrow \mathcal{M}$  is equivalent (by adjunction) to the data of a diagram  $\mathcal{F} : \mathcal{J}_{/i}^+ \longrightarrow \mathcal{M}_{\mathcal{F}(*)/}$ . It follows that

$$L_i^{\mathcal{J}_*}(\mathcal{F}) = \text{colim}_{j \rightarrow i \in \mathcal{J}_{*/i}^+} \mathcal{F}(j) = \left[ \text{colim}_{j \rightarrow i \in \mathcal{J}_{/i}^+} \mathcal{F}(j) \right]_{\text{colim}_{j \rightarrow i \in \mathcal{J}_{/i}^+} \mathcal{F}(*)} \coprod_{\text{colim}_{j \rightarrow i \in \mathcal{J}_{/i}^+} \mathcal{F}(*)} \mathcal{F}(*) = L_i^{\mathcal{J}}(\mathcal{F}|_{\mathcal{J}}) \coprod_{L_i^{\mathcal{J}}(\mathcal{F}(*))} \mathcal{F}(*)$$

and similarly

$$L_i^{\mathcal{J}_*}(\mathcal{G}) = L_i^{\mathcal{J}}(\mathcal{G}|_{\mathcal{J}}) \coprod_{L_i^{\mathcal{J}}(\mathcal{G}(*))} \mathcal{G}(*)$$

where by abuse of notation we considered  $\mathcal{F}(*)$  and  $\mathcal{G}(*)$  as constant functors  $\mathcal{J}_+ \longrightarrow \mathcal{M}$ . We now



see that both 3.3.1 and 3.3.2 can be identified with the colimit of the diagram

$$\begin{array}{ccccc}
 \mathcal{F}(i) & \longleftarrow & L_i^j(\mathcal{F}|_j) & \xrightarrow{\text{Id}} & L_i^j(\mathcal{F}|_j) \\
 \uparrow & & \uparrow & & \uparrow \text{Id} \\
 \mathcal{F}(*) & \longleftarrow & L_i^j(\mathcal{F}(*)) & \longrightarrow & L_i^j(\mathcal{F}|_j) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{G}(*) & \longleftarrow & L_i^j(\mathcal{G}(*)) & \longrightarrow & L_i^j(\mathcal{G}|_j)
 \end{array}$$

in the category  $\mathcal{M}_{/\mathcal{G}(i)}$ : for 3.3.1 we first compute the rows and for 3.3.2 we start with the columns, using that  $L_i^j$  preserves colimits for the middle column.  $\square$

Now let  $\mathcal{M}$  be a proper combinatorial model category. Since  $\mathbb{N} \times \mathbb{N}$  is a Reedy category with only increasing maps, the Reedy and the projective model structures on  $(\mathcal{M}_{A//A})^{\mathbb{N} \times \mathbb{N}}$  agree. We now claim that  $\mathcal{T}\mathcal{M}$  can be viewed as a left Bousfield localization of  $\int_{\mathcal{M}} (\mathcal{M}_{A//A})_{\text{proj}}^{\mathbb{N} \times \mathbb{N}} \cong \int_{\mathcal{M}} (\mathcal{M}_{A//A})_{\text{Reedy}}^{\mathbb{N} \times \mathbb{N}}$ . Indeed, under the equivalence of Lemma 3.3.1 it suffices to identify  $\mathcal{T}\mathcal{M}$  with a left Bousfield localization of the Reedy functor category  $\mathcal{M}_{\text{Reedy}}^{(\mathbb{N} \times \mathbb{N})*}$ . To describe a set of maps inducing this left Bousfield localization, observe that a Reedy fibrant object  $X_* \rightarrow X_{\bullet\bullet} \rightarrow X_*$  in  $\mathcal{M}^{(\mathbb{N} \times \mathbb{N})*}$  is fibrant in  $\mathcal{T}\mathcal{M}$  if and only if the map  $X_{n,m} \rightarrow X_*$  is a weak equivalence for every  $n \neq m$  and for every  $n \geq 0$  the square

$$\begin{array}{ccc}
 X_{n,n} & \longrightarrow & X_{n+1,n} \\
 \downarrow & & \downarrow \\
 X_{n+1,n} & \longrightarrow & X_{n+1,n+1}
 \end{array} \tag{3.3.3}$$

is homotopy Cartesian (indeed, since  $X_*$  is fibrant this is the same as saying that 3.3.3 is homotopy Cartesian when considered as a square in  $\mathcal{M}_{X_*//X_*}$ ). Since  $\mathcal{M}$  is combinatorial there exists a set of objects  $\mathcal{D}$  such that a map  $f : A \rightarrow B$  in  $\mathcal{M}$  is a weak equivalence if and only if  $\text{Map}_{\mathcal{M}}^h(D, A) \rightarrow \text{Map}_{\mathcal{M}}^h(D, B)$  is a weak equivalence of spaces for every  $D \in \mathcal{D}$ . We may now define  $S_{\mathcal{T}\mathcal{M}}$  to be the set of maps

$$h_* \otimes D \rightarrow h_{n,m} \otimes D \quad n \neq m, D \in \mathcal{D}$$

together with the maps

$$\left[ h_{n+1,n} \coprod_{h_{n+1,n+1}} h_{n,n+1} \right] \otimes D \rightarrow h_{n,n} \otimes D \quad n \geq 0, D \in \mathcal{D}$$

where for  $x \in (\mathbb{N} \times \mathbb{N})_*$  we denote by  $h_x : (\mathbb{N} \times \mathbb{N})_* \rightarrow \text{Set}$  the corresponding corepresentable functor and  $\otimes$  denotes the natural tensoring of  $\mathcal{M}$  over sets. Arguing as in the proof of Lemma 2.1.6 we see that the integral model structure on  $\mathcal{T}\mathcal{M}$  is the left Bousfield localization of  $\mathcal{M}_{\text{Reedy}}^{(\mathbb{N} \times \mathbb{N})*}$  with respect to  $S_{\mathcal{T}\mathcal{M}}$ .

*Remark 3.3.2.* The above remarks allow one to define the tangent bundle  $\mathcal{T}\mathcal{M}$  for a left proper model category  $\mathcal{M}$  which is not necessarily right proper. Indeed, one just defines  $\mathcal{T}\mathcal{M}$  as the left Bousfield localization of  $\mathcal{M}_{\text{Reedy}}^{(\mathbb{N} \times \mathbb{N})*}$  with respect to the set of maps  $S_{\mathcal{T}\mathcal{M}}$ . The resulting model category comes with a natural left and right Quillen functor  $\pi : \mathcal{T}\mathcal{M} \rightarrow \mathcal{M}$ , evaluating on  $*$ .

Although  $\pi$  is not always a model fibration when  $\mathcal{M}$  is not right proper the induced map of  $\infty$ -categories  $\pi_{\infty} : \mathcal{T}\mathcal{M}_{\infty} \rightarrow \mathcal{M}_{\infty}$  still exhibits  $\mathcal{T}\mathcal{M}_{\infty}$  as a tangent bundle to  $\mathcal{M}_{\infty}$  (in the sense of

Proposition 3.2.6). Indeed, let  $j : [1] \rightarrow (\mathbb{N} \times \mathbb{N})_*$  be the inclusion of the arrow  $(0, 0) \rightarrow *$  in  $(\mathbb{N} \times \mathbb{N})_*$ . Restriction along  $j$  induces a diagram of right Quillen functors

$$\begin{array}{ccc} \mathcal{T}\mathcal{M} & \xrightarrow{j^*} & \mathcal{M}_{\text{proj}}^{[1]} \\ & \searrow \pi & \swarrow \text{ev}_1 \\ & \mathcal{M}. & \end{array}$$

This induces a triangle of  $\infty$ -categories of the form (3.2.1). To see that this realizes  $\mathcal{T}\mathcal{M}_\infty$  as the tangent bundle to  $\mathcal{M}_\infty$ , let  $\mathcal{T}\mathcal{M}' \subseteq \mathcal{T}\mathcal{M}$  be the full relative subcategory on those fibrant objects in  $\mathcal{T}\mathcal{M}$  whose image under  $\pi$  is fibrant-cofibrant and let  $\mathcal{M}'^{[1]} \subseteq \mathcal{M}^{[1]}$  be the full subcategory of fibrations with fibrant-cofibrant codomain. Both of these inclusions are equivalences of relative categories, with homotopy inverse given by fibrant-cofibrant replacement. Now observe that  $\mathcal{T}\mathcal{M}'$  is the Grothendieck construction of the relative functor  $\mathcal{T}_{(-)}\mathcal{M} : (\mathcal{M}^{\text{fib-cof}})^{\text{op}} \rightarrow \text{RelCat}$ , associating to each fibrant object in  $\mathcal{M}$  the relative category  $\mathcal{T}_A\mathcal{M}^{\text{fib}}$ . Similarly  $\mathcal{M}'$  is obtained by integrating  $A \mapsto (\mathcal{M}_{/A})^{\text{fib}}$  over all fibrant-cofibrant  $A$  and  $j^* : \mathcal{T}\mathcal{M}' \rightarrow \mathcal{M}'^{[1]}$  is obtained by integrating the natural relative functor  $\Omega_+^\infty : \mathcal{T}_A\mathcal{M}^{\text{fib}} \rightarrow (\mathcal{M}_{/A})^{\text{fib}}$ . The result then follows from Proposition 2.2.5 and [Hin13, Proposition 2.1.4].

Now suppose that  $\mathcal{M}$  is tensored over a combinatorial SM model category  $\mathbf{S}$ , so that  $\mathcal{M}^{(\mathbb{N} \times \mathbb{N})_*}$  inherits a natural levelwise tensor structure (see [Bar07]). The following proposition shows that if  $\mathbf{S}$  is **tractable**, i.e., combinatorial and such that the domains of the generating (trivial) cofibrations are cofibrant, then there is a simple criterion for determining when a tensor structure over  $\mathbf{S}$  descends to a given left Bousfield localization.

**PROPOSITION 3.3.3.** *Let  $\mathbf{S}$  be a SM tractable model category with generating cofibrations  $I = \{K_\alpha \rightarrow L_\alpha\}$  and  $\mathcal{M}$  a left proper combinatorial model category which is tensored and cotensored over  $\mathbf{S}$ . Let  $L_S\mathcal{M}$  be a left Bousfield localization of  $\mathcal{M}$  at a set of maps  $S$  between cofibrant objects. If cotensoring with a cofibrant object in  $\mathbf{S}$  preserves  $S$ -local objects in  $\mathcal{M}$  then the tensor-cotensor structure of  $\mathcal{M}$  over  $S$  is compatible with the localized model structure. In particular,  $L_S\mathcal{M}$  inherits a tensor-cotensor structure over  $\mathbf{S}$ .*

*Proof.* It is enough to check that the pushout-product of a map  $i : K_\alpha \rightarrow L_\alpha$  in  $I$  against a trivial cofibration  $X \rightarrow Y$  in  $L_S\mathcal{M}$  is a local weak equivalence. If cotensoring with a cofibrant object  $K$  in  $\mathbf{S}$  preserves  $S$ -local objects in  $\mathcal{M}$ , then the Quillen adjunction  $K \otimes (-) : \mathcal{M} \rightleftarrows \mathcal{M} : (-)^K$  descends to a Quillen pair  $K \otimes (-) : L_S\mathcal{M} \rightleftarrows L_S\mathcal{M} : (-)^K$  by Lemma 3.1.2. Since the objects  $K_\alpha$  and  $L_\alpha$  are cofibrant by tractability of  $\mathbf{S}$ , the maps  $K_\alpha \otimes X \rightarrow K_\alpha \otimes Y$  and  $L_\alpha \otimes X \rightarrow L_\alpha \otimes Y$  are trivial cofibrations in  $L_S\mathcal{M}$ . Since the cobase change of a trivial cofibration is again a trivial cofibration, it follows from the 2-out-of-3 property in  $L_S\mathcal{M}$  that the pushout-product map

$$K_\alpha \otimes Y \coprod_{K_\alpha \otimes X} L_\alpha \otimes X \rightarrow L_\alpha \otimes Y$$

is a weak equivalence in  $L_S\mathcal{M}$ . □

**COROLLARY 3.3.4.** *Let  $\mathcal{M}$  be a combinatorial proper model category which is tensored and cotensored over a tractable SM model category  $\mathbf{S}$ . Then  $\mathcal{T}\mathcal{M}$  is naturally tensored and cotensored over  $\mathbf{S}$ , where the tensor structure is given by*

$$K \otimes (B \rightarrow X_{\bullet\bullet} \rightarrow B) = K \otimes B \rightarrow K \otimes X_{\bullet\bullet} \rightarrow K \otimes B$$

*and the cotensor is given by*

$$(B \rightarrow X_{\bullet\bullet} \rightarrow B)^K = B^K \rightarrow (X_{\bullet\bullet})^K \rightarrow B^K.$$

*Proof.* By [Bar07, Lemma 4.2] the levelwise tensor-cotensor structure over  $\mathbf{S}$  is compatible with the Reedy model structure on  $\mathcal{M}^{(\mathbb{N} \times \mathbb{N})^*}$ . To verify the condition of Proposition 3.3.3, it suffices to prove that cotensoring with a cofibrant object  $K \in \mathbf{S}$  preserves fibrant objects in  $\mathcal{TM}$ , i.e.  $\Omega$ -spectra. But this follows from the fact that cotensoring with  $K$  preserves weak equivalences between fibrant objects and homotopy Cartesian squares involving fibrant objects, since  $(-)^K : \mathcal{M} \rightarrow \mathcal{M}$  is a right Quillen functor.  $\square$

EXAMPLE 3.3.5. If  $\mathcal{M}$  is a simplicial combinatorial proper model category then  $\mathcal{TM}$  is naturally a simplicial model category.

EXAMPLE 3.3.6. If  $\mathcal{M}$  is an SM tractable proper model category then  $\mathcal{TM}$  is naturally tensored over  $\mathcal{M}$ .

### 3.4 Tangent bundles of functor categories

Let  $\mathbf{S}$  be tractable SM model category and let  $\mathcal{M}$  be a left proper combinatorial model category tensored over  $\mathbf{S}$ . As described in the previous section, if  $\mathcal{M}$  is also right proper then we can construct the tangent bundle  $\mathcal{TM}$  model category using the machinery of [HP15], in which case the resulting category can be identified with a suitable left Bousfield localization of a Reedy functor category of the form  $\mathcal{M}^{(\mathbb{N} \times \mathbb{N})^*}$ . As explained in Remark 3.3.2, when  $\mathcal{M}$  is not right proper we may simply define  $\mathcal{TM}$  to be the above functor category. The canonical projection  $\mathcal{TM} \rightarrow \mathcal{M}$  is then not necessarily a model fibration, but is still a model for the  $\infty$ -categorical tangent bundle.

Let  $\mathcal{J}$  be a small  $\mathbf{S}$ -enriched category. By Corollary 3.3.4 the model category  $\mathcal{TM}$  inherits a natural  $\mathbf{S}$ -enrichment, and we may hence consider the category  $\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{TM})$  of  $\mathbf{S}$ -enriched functors  $\mathcal{J} \rightarrow \mathcal{TM}$ . Unless otherwise stated we will consider categories of enriched functors as endowed with the projective model structure. We then have the following proposition:

PROPOSITION 3.4.1. *We have a natural equivalence of categories over  $\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$ :*

$$\mathcal{T}\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M}) \cong \text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{TM}). \quad (3.4.1)$$

*In other words, the tangent bundle of a functor category into  $\mathcal{M}$  (endowed with the projective model structure) is the category of functors into the tangent bundle of  $\mathcal{M}$  (endowed with the projective model structure).*

*Proof.* By Lemma 3.3.1 and the discussion following it we may identify the left hand side of 3.4.1 with suitable left Bousfield localizations of the iterated projective model structure on  $\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})^{(\mathbb{N} \times \mathbb{N})^*}$ , and the right hand side with a suitable left Bousfield localization of the iterated projective model structure on  $\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M}^{(\mathbb{N} \times \mathbb{N})^*})$ . Both of these categories (before localization) can be identified with the category of enriched functors  $\mathcal{J} \otimes (\mathbb{N} \times \mathbb{N})_+ \rightarrow \mathcal{M}$ , endowed with the projective model structure. Here the tensor product of an enriched category by a discrete category is given by the Cartesian product on object sets and by the natural tensoring of  $\mathbf{S}$  over sets on mapping objects. Under this identification we now see that the two left Bousfield localizations coincide. Indeed, a levelwise fibrant enriched functor  $\mathcal{F} : \mathcal{J} \otimes (\mathbb{N} \times \mathbb{N})_+ \rightarrow \mathcal{M}$  is local in either the left or the right hand side of 3.4.1 if and only if for every  $i \in \mathcal{J}$  the restriction  $\mathcal{F}|_{i \times (\mathbb{N} \times \mathbb{N})}$  is an  $\Omega$ -spectrum object of  $\mathcal{M}_{\mathcal{F}(i, *) / \mathcal{F}(i, *)}$ .  $\square$

Remark 3.4.2. Since the equivalence 3.4.1 is an equivalence over  $\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$  we obtain for every  $\mathcal{F} : \mathcal{J} \rightarrow \mathcal{M}$  an induced equivalence of categories

$$\text{Sp}(\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})_{\mathcal{F} // \mathcal{F}}) \xrightarrow{\cong} \text{Fun}_{/\mathcal{M}}^{\mathbf{S}}(\mathcal{J}, \mathcal{TM}) \quad (3.4.2)$$

where  $\text{Fun}_{/\mathcal{M}}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$  denotes the category  $\mathbf{S}$ -enriched lifts

$$\begin{array}{ccc} & & \mathcal{M} \\ & \nearrow & \downarrow \pi \\ \mathcal{J} & \xrightarrow{\mathcal{F}} & \mathcal{M} \end{array}$$

By transport of structure one obtains a natural model structure  $\text{Fun}_{/\mathcal{M}}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$ , which coincides, in this case, with the corresponding projective model structure (i.e., where weak equivalences and fibrations are defined objectwise).

When  $\mathcal{M}$  is furthermore stable the situation becomes even simpler. Indeed, in this case  $\text{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$  is stable and is Quillen equivalent to both sides of (3.4.2) under mild assumptions. This follows from Corollary 2.2.7 and the following lemma:

**LEMMA 3.4.3.** *Let  $\mathcal{M}$  be a stable model category equipped with a strict zero object  $0 \in \mathcal{M}$  and let  $A \in \mathcal{M}$  be an object. Assume that either  $A$  is cofibrant or  $\mathcal{M}$  is left proper and that either  $A$  is fibrant or  $\mathcal{M}$  is right proper. Then the adjunction*

$$(-) \coprod A : \mathcal{M} \xrightleftharpoons{\quad} \mathcal{M}_{A//A} : \ker$$

*induced by applying Construction 3.1.4 twice to the map  $0 \rightarrow A$  is a Quillen equivalence.*

*Proof.* The functor  $\ker$  sends an object  $A \rightarrow C \xrightarrow{p} A$  over under  $A$  to the object  $\ker(p) = C \times_A 0$ , while its left adjoint sends an object  $B$  to the object  $A \rightarrow B \coprod A \rightarrow A$ , where the first map is the inclusion of the second factor and the second map restricts to the identity on  $A$  and to the 0-map on  $B$ .

Let  $B \in \mathcal{M}$  be a cofibrant object and  $A \rightarrow C \xrightarrow{p} A$  a fibrant object of  $\mathcal{M}_{A//A}$ . We have to show that a map  $f : B \coprod A \rightarrow C$  over-under  $A$  is a weak equivalence if and only if the adjoint map  $f^{\text{ad}} : B \rightarrow C \times_A 0$  is a weak equivalence. These two maps fit into a diagram in  $\mathcal{M}$  of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{f^{\text{ad}}} & C \times_A 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B \coprod A & \xrightarrow{f} & C & \xrightarrow{p} & A \end{array}$$

where the left square is coCartesian and the right square is Cartesian. Under the assumption that  $A$  is cofibrant or  $\mathcal{M}$  is left proper the left square is homotopy coCartesian. Under the assumption that  $A$  is fibrant or  $\mathcal{M}$  is right proper the right square is homotopy Cartesian. Since the external rectangle is clearly homotopy Cartesian and coCartesian and since  $\mathcal{M}$  is stable, it follows from Remark 2.2.2 and the pasting lemma for homotopy (co)Cartesian squares that all squares in this diagram are homotopy Cartesian and coCartesian. This means in particular that the top middle horizontal map is an equivalence iff the bottom middle horizontal map is one.  $\square$

**COROLLARY 3.4.4.** *Let  $\mathcal{M}$  be a proper combinatorial strictly pointed stable model category. Then  $\mathcal{M}_{A//A}$  is stable as well, and we may identify  $\Sigma_+^{\infty}(A) \in \text{Sp}(\mathcal{M}_{A//A}) \simeq \mathcal{M}_{A//A}$  with  $A \coprod A$ . In particular, the image of  $\Sigma_+^{\infty}(A)$  under the composed equivalence  $\text{Sp}(\mathcal{M}_{A//A}) \xrightarrow{\sim} \mathcal{M}_{A//A} \xrightarrow{\sim} \mathcal{M}$  is just  $A$  itself.*

**COROLLARY 3.4.5.** *Let  $\mathcal{M}$  be a proper combinatorial strictly pointed stable model category. Then*

the Quillen equivalences of Lemma 3.4.3 assemble to a Quillen equivalence

$$\mathcal{M} \times \mathrm{Sp}(\mathcal{M}) \xrightarrow[\leftarrow]{\simeq} \mathcal{T}\mathcal{M}.$$

In particular,  $\mathcal{T}\mathcal{M}$  is Quillen equivalent to  $\mathcal{M} \times \mathcal{M}$ .

*Proof.* Apply [HP15, Theorem 4.1.3] and Corollary 2.2.7.  $\square$

**COROLLARY 3.4.6.** *Let  $\mathcal{M}$  be a proper combinatorial strictly pointed stable model category tensored over a tractable SM model category  $\mathbf{S}$ . Then for every  $\mathbf{S}$ -enriched functor  $\mathcal{F} : \mathcal{J} \longrightarrow \mathcal{M}$  the tangent model category  $\mathcal{T}_{\mathcal{F}} \mathrm{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$  is Quillen equivalent to  $\mathrm{Fun}^{\mathbf{S}}(\mathcal{J}, \mathcal{M})$ .*

#### 4. Stabilization of algebras over operads

In this section we will turn our attention to the case of algebras over colored operads and establish the main results of this paper, as described in the introduction. We will begin in §4.1 by recalling preliminaries and establishing notation concerning colored symmetric operads and their algebras. In §4.2 we will prove the main core results, relating the the stabilization of the category of augmented algebras over an operad  $\mathcal{P}$  to the stabilization of the category of algebras over a suitable 1-skeleton  $\mathcal{P}_{\leq 1}$  of  $\mathcal{P}$ . Our proof makes use of a well-known filtration on free algebras, but requires a somewhat detailed variant thereof, which we develop in the appendix A. We then show in §4.3 how this comparison result can be used to equate the tangent categories of algebras with tangent categories of modules. The latter can then be described explicitly as suitable categories of enriched lifts, using §3.4. In the last section §4.4 we show how to harness the results of §4.2 to obtain analogous results in the  $\infty$ -categorical setting.

##### 4.1 Preliminaries on colored operads

Throughout this section, let  $\mathcal{M}$  be a symmetric monoidal (SM) locally presentable category in which the tensor product distributes over colimits.

**DEFINITION 4.1.1.** Let  $\Sigma$  be the skeleton of the groupoid of finite sets and bijections between them consisting of the sets  $\underline{n} = \{1, \dots, n\}$  for every  $n$  (where  $\underline{0} = \emptyset$  by convention). In particular, the automorphism group  $\mathrm{Aut}(\underline{n})$  can be identified with the symmetric group on  $n$  elements. For every  $\underline{n}$  we will denote by  $\underline{n}_+ = \underline{n} \amalg \{*\} = \{*, 1, \dots, n\}$ . We consider the association  $\underline{n} \mapsto \underline{n}_+$  as a functor  $\iota : \Sigma \longrightarrow \mathrm{Set}$ .

For a set  $W$  we will denote by  $\Sigma_W = \Sigma \times_{\mathrm{Set}} \mathrm{Set}/_W$  the comma category associated to  $\iota$ . More explicitly, we may identify objects of  $\Sigma_W$  with pairs  $(\underline{n}, \overline{w})$  where  $\underline{n}$  is an object of  $\Sigma$  and  $\overline{w} : \underline{n}_+ \longrightarrow W$  is a map of sets. We think of  $\overline{w}$  as a vector of elements of  $W$  indexed  $\underline{n}_+$  and will denote the individual components by  $w_*, w_1, \dots, w_n$ . We will refer to  $n$  as the **arity** of the object  $\overline{w}$ . We will generally abuse notation and refer to the object  $(\underline{n}, \overline{w})$  simply by  $\overline{w}$ , suppressing the explicit reference to the arity. We note that  $\Sigma_W$  is a groupoid and denote the automorphism group of  $\overline{w}$  by  $\mathrm{Aut}(\overline{w})$ . If  $\overline{w}$  has arity  $n$  then  $\mathrm{Aut}(\overline{w})$  can be identified with the subgroup of  $\mathrm{Aut}(\underline{n})$  consisting of those permutations  $\sigma$  such that  $\overline{w} \circ \sigma = \overline{w}$ .

**DEFINITION 4.1.2.** A  **$W$ -symmetric sequence** in  $\mathcal{M}$  is a functor  $X : \Sigma_W \longrightarrow \mathcal{M}$ . We will denote by  $\mathrm{SymSeq}_W(\mathcal{M})$  the category of  $W$ -symmetric sequences.

Recall that the category  $\mathrm{SymSeq}_W(\mathcal{M})$  admits a (non-symmetric) monoidal product known as the **composition product**, which can be described as follows: consider the groupoid  $\mathrm{Ar}$

whose objects are (not necessarily bijective) maps of finite sets  $\phi : \underline{k} \rightarrow \underline{n}$  and whose morphisms are natural bijections between such maps. There is a functor  $\text{dom} : \text{Ar} \rightarrow \text{Set}$  sending  $\phi : \underline{k} \rightarrow \underline{n}$  to  $\underline{k}_+$  and a functor  $\text{sum} : \text{Ar} \rightarrow \text{Set}$  sending  $\phi : \underline{k} \rightarrow \underline{n}$  to  $(\underline{k} \amalg \underline{n})_+$ .

Let  $\text{Dec}_W = \text{Ar} \times_{\text{Set}} \text{Set}/_W$  be the comma category associated to the functor  $\text{sum}$ . Explicitly, the objects of  $\text{Dec}_W$  are given by tuples  $(\phi, \bar{v})$  consisting of a map of finite sets  $\phi : \underline{k} \rightarrow \underline{n}$  and a map  $\bar{v} : (\underline{k} \amalg \underline{n})_+ \rightarrow W$ . We will denote by  $\phi_+ : \underline{k} \amalg \underline{n} \rightarrow \underline{n}$  the map which restricts to  $\phi$  on  $\underline{k}$  and to the identity on  $\underline{n}$ . We note that the identity  $\underline{n} \rightarrow \underline{n}$  induces a natural section of  $\phi_+$ , and so we may consider  $\phi_+$  as a pointed object of  $\text{Set}/_{\underline{n}}$ . In particular, for every  $i = 1, \dots, n$  the inverse image  $\phi_+^{-1}(i) = \phi^{-1}(i) \cup \{i\}$  is naturally a pointed set with base point  $i$ . If  $\bar{v} : (\underline{k} \amalg \underline{n})_+ \rightarrow W$  is an object of  $\Sigma_W$  then we may consider  $\bar{v}|_{\phi_+^{-1}(i)} : \phi_+^{-1}(i) \rightarrow W$  as a map from the pointification of  $\phi^{-1}(i)$  to  $W$ . Since  $\Sigma$  is a skeleton of the category of finite sets and bijection we can consider the association  $(\phi : \underline{k} \rightarrow \underline{n}, \bar{v}) \mapsto (\phi^{-1}(i), \bar{v}|_{\phi_+^{-1}(i)})$  as a functor  $\text{Dec}_W \rightarrow \Sigma_W$ .

If  $X$  and  $Y$  are two  $W$ -symmetric sequences, one defines their composition product by  $X \circ Y = \text{Lan}_\pi(X \boxtimes Y)$  where  $X \boxtimes Y : \text{Dec}_W \rightarrow \mathcal{M}$  is given by

$$(\phi : \underline{k} \rightarrow \underline{n}, \bar{v}) \mapsto X(\bar{v}|_{\underline{n}_+}) \otimes \bigotimes_{i \in \underline{n}} Y(\bar{v}|_{\phi_+^{-1}(i)})$$

Explicitly, for  $\bar{w}$  of arity  $k$ , the composition product is given by the formula:

$$(X \circ Y)(\bar{w}) = \coprod_{[(\phi, \bar{v})]} \left[ X(\bar{v}|_{\underline{n}_+}) \otimes \bigotimes_{i \in \underline{n}} Y(\bar{v}|_{\phi_+^{-1}(i)}) \right] \otimes_{\text{Aut}(\phi, \bar{v})} \text{Aut}(\bar{w}) \quad (4.1.1)$$

where the coproduct runs over all isomorphism classes of objects  $(\phi : \underline{k} \rightarrow \underline{n}, \bar{v} : (\underline{k} \amalg \underline{n})_+ \rightarrow W) \in \text{Dec}_W$  such that  $\bar{v}|_{\underline{k}_+} = \bar{w}$ , while  $\text{Aut}(\phi, v)$  is the automorphism group  $(\phi, v)$  in  $\text{Dec}_W$ . We refer the reader to [PS14] for more details on the composition product (which is called the “substitution product” in loc.cit.).

**DEFINITION 4.1.3.** A  $W$ -colored **(symmetric) operad**  $\mathcal{P}$  is a monoid object in  $\text{SymSeq}_W(\mathcal{M})$  with respect to the composition product described above. We will usually not mention the term “symmetric” explicitly when discussing such operads, and will omit the term “ $W$ -colored” whenever  $W$  is clear in the context. We will denote by  $\text{Op}_W(\mathcal{M})$  the category of  $W$ -colored operads in  $\mathcal{M}$ .

Explicitly, a  $W$ -colored operad  $\mathcal{P}$  consists of objects  $\mathcal{P}(\bar{w})$ , considered as parametrizing  $n$ -ary **operations from**  $w_1, \dots, w_n$  **to**  $w_*$ , and for every  $\phi : \underline{k} \rightarrow \underline{n}$  and  $\bar{v} : (\underline{k} \amalg \underline{n})_+ \rightarrow W$  as above, a composition operation

$$\mathcal{P}(\bar{v}|_{\underline{n}_+}) \otimes \bigotimes_{i \in \underline{n}} \mathcal{P}(\bar{v}|_{\phi_+^{-1}(i)}) \rightarrow \mathcal{P}(\bar{v}|_{\underline{k}_+}),$$

subject to the natural equivariance, associativity and unitality conditions.

**DEFINITION 4.1.4.** Let  $\mathcal{P}$  be a  $W$ -colored operad in  $\mathcal{M}$ . A left (resp. right) **module** over  $\mathcal{P}$  is a  $W$ -symmetric sequence in  $\mathcal{M}$  which is a left (resp. right) module over  $\mathcal{P}$  with respect to the composition product above. A  $\mathcal{P}$ -**algebra** is a left  $\mathcal{P}$ -module  $A \in \text{SymSeq}_W(\mathcal{M})$  which is concentrated in arity 0, i.e., such that  $A(\bar{w}) = \emptyset_{\mathcal{M}}$  whenever  $\bar{w}$  is of arity  $n > 0$ .

Explicitly, a  $\mathcal{P}$ -algebra is given by an object  $A \in \mathcal{M}^W$ , together with maps

$$\mathcal{P}(\bar{w}) \otimes A(w_1) \otimes \dots \otimes A(w_n) \rightarrow A(w_*)$$

subject to the natural equivariance, associativity and unitality constraints. We denote by  $\text{Alg}_{\mathcal{P}}(\mathcal{M})$  the category of  $\mathcal{P}$ -algebras and algebra maps. When there is no possibility of confusion we will also denote  $\text{Alg}_{\mathcal{P}}(\mathcal{M})$  simply by  $\text{Alg}_{\mathcal{P}}$ .

The groupoid  $\Sigma_W$  can be decomposed as a disjoint union  $\Sigma_W \cong \coprod_{n \geq 0} \Sigma_W^n$  where  $\Sigma_W^n$  is the full subgroupoid consisting of objects of arity  $n$ . Let  $j_n : \Sigma_W^n \rightarrow \Sigma_W$  be the inclusion of  $\Sigma_W^n$  and  $t_n : \Sigma_W^{\leq n} \rightarrow \Sigma_W$  the inclusion of  $\coprod_{m \leq n} \Sigma_W^m$ .

**DEFINITION 4.1.5.** Let  $\mathcal{P}$  be a  $W$ -colored symmetric sequence in  $\mathcal{M}$ . We define the **arity  $n$  part** of  $\mathcal{P}$  to be the  $W$ -symmetric sequence  $\mathcal{P}_n := \text{Lan}_{j_n} j_n^* \mathcal{P}$  and the  **$n$ -skeleton** of  $\mathcal{P}$  to be the  $W$ -symmetric sequence  $\mathcal{P}_{\leq n} := \text{Lan}_{t_n} t_n^* \mathcal{P}$ . When  $n = 0$ , we denote by  $\mathcal{P}_0^+$  the free  $W$ -colored operad generated from the  $W$ -symmetric sequence  $\mathcal{P}_0 = \mathcal{P}_{\leq 0}$ .

Explicitly, the symmetric sequence  $\mathcal{P}_n$  (resp.  $\mathcal{P}_{\leq n}$ ) is given by  $\mathcal{P}_n(\overline{w}) = \mathcal{P}(\overline{w})$  for  $\overline{w}$  of arity  $n$  (resp.  $\text{arity} \leq n$ ) and  $\mathcal{P}_n(\overline{w}) = \emptyset$  for  $\overline{w}$  of arity  $\neq n$  (resp.  $\text{arity} > n$ ). The operad  $\mathcal{P}_0^+$  has no non-trivial  $m$ -ary operations for  $m > 1$  (i.e., the corresponding objects of  $m$ -ary operations are all initial), while  $\mathcal{P}_0^+(\overline{w}) = \mathcal{P}_0(\overline{w})$  for  $\overline{w}$  of arity 0 and its 1-ary operations are only identity maps.

Let  $\mathcal{P}$  be a  $W$ -colored operad. Then  $\mathcal{P}_{\leq 1}$  and  $\mathcal{P}_1$  inherit from  $\mathcal{P}$  a natural operad structure. Furthermore,  $\mathcal{P}_n$  inherits from  $\mathcal{P}$  the structure of a  $\mathcal{P}_1$ -bimodule and  $\mathcal{P}_{\leq n}$  inherits from  $\mathcal{P}$  the structure of a  $\mathcal{P}_{\leq 1}$ -bimodule. Similarly,  $\mathcal{P}_0 = \mathcal{P}_{\leq 0}$  inherits from  $\mathcal{P}$  the structure of a  $\mathcal{P}$ -bimodule, and is in particular a  $\mathcal{P}$ -algebra. As such, it is the **initial  $\mathcal{P}$ -algebra**.

An **augmented  $\mathcal{P}$ -algebra** in  $\mathcal{M}$  is a  $\mathcal{P}$ -algebra  $A$  equipped with a map of  $\mathcal{P}$ -algebras  $A \rightarrow \mathcal{P}_0$ , where  $\mathcal{P}_0$  is considered as the initial  $\mathcal{P}$ -algebra. We will denote by  $\text{Alg}_{\mathcal{P}}^{\text{aug}} = (\text{Alg}_{\mathcal{P}})_{/\mathcal{P}_0}$  the category of augmented  $\mathcal{P}$ -algebras. We note that by construction the category  $\text{Alg}_{\mathcal{P}}^{\text{aug}}$  is pointed.

**EXAMPLE 4.1.6.** A  $W$ -colored operad in  $\mathcal{M}$  with only 1-ary operations is precisely an  $\mathcal{M}$ -enriched category with  $W$  as its set of objects. Consequently, if  $\mathcal{P}$  is an operad in  $\mathcal{M}$  then we will often consider  $\mathcal{P}_1$  as an  $\mathcal{M}$ -enriched category, and will refer to it as the **underlying category** of  $\mathcal{P}$ . When  $\mathcal{P}$  is an  $\mathcal{M}$ -enriched category (i.e., when  $\mathcal{P} = \mathcal{P}_1$ ), a  $\mathcal{P}$ -algebra is simply an enriched functor  $\mathcal{P} \rightarrow \mathcal{M}$ .

Every morphism of  $W$ -coloured operads  $f : \mathcal{P} \rightarrow \mathcal{Q}$  induces an extension-restriction adjunction

$$f_! : \text{Alg}_{\mathcal{P}} \rightleftarrows \text{Alg}_{\mathcal{Q}} : f^*.$$

Let  $\int_{\mathcal{P} \in \text{Op}} \text{Alg}_{\mathcal{P}}$  be the Grothendieck construction of the functor  $\mathcal{P} \mapsto \text{Alg}_{\mathcal{P}}$  and  $f \mapsto f_!$ . As in [BM09, Definition 1.5], one may consider the functor

$$\text{Op}_W \rightarrow \int_{\mathcal{P} \in \text{Op}} \text{Alg}_{\mathcal{P}}$$

sending a  $W$ -colored operad  $\mathcal{P}$  to the pair  $(\mathcal{P}, \mathcal{P}_0)$  consisting of  $\mathcal{P}$  and its initial  $\mathcal{P}$ -algebra. This functor admits a left adjoint

$$\text{Env} : \int_{\mathcal{P} \in \text{Op}_W} \text{Alg}_{\mathcal{P}} \rightarrow \text{Op}_W$$

associating to a pair  $(\mathcal{P}, A)$  of an operad  $\mathcal{P}$  and a  $\mathcal{P}$ -algebra  $A$  a new operad  $\mathcal{P}^A \stackrel{\text{def}}{=} \text{Env}(\mathcal{P}, A) \in \text{Op}_W$ . Following [BM09] we will refer to  $\mathcal{P}^A$  as the **enveloping operad** of  $A$ , and refer to the  $\mathcal{M}$ -enriched category  $\mathcal{P}_1^A$  as the **enveloping category** of  $A$ . The category of algebras over  $\mathcal{P}^A$  is equivalent to the category  $(\text{Alg}_{\mathcal{P}})_{A/}$  of  $\mathcal{P}$ -algebras under  $A$  (see [PS14, Proposition 4.4(iv)]). When  $A = \mathcal{P}_0$  is the initial  $\mathcal{P}$ -algebra the natural map  $\mathcal{P} \rightarrow \mathcal{P}^A$  is an isomorphism ([PS14, Proposition 4.4(i)]).

DEFINITION 4.1.7. Let  $\mathcal{P}$  be an operad and  $A$  a  $\mathcal{P}$ -algebra. An  $A$ -**module** is an algebra over  $\mathcal{P}_1^A$ , i.e., an  $\mathcal{M}$ -enriched functor from the enveloping category of  $A$  to  $\mathcal{M}$ . We will denote by  $\text{Mod}_A^{\mathcal{P}}(\mathcal{M})$  the category of  $A$ -modules in  $\mathcal{M}$ . When there is no possibility of confusion we will also denote  $\text{Mod}_A^{\mathcal{P}}(\mathcal{M})$  simply by  $\text{Mod}_A^{\mathcal{P}}$ .

Unwinding the definition, one find that a module over a  $\mathcal{P}$ -algebra  $A$  is given by an object  $M \in \mathcal{M}^W$  together with action maps

$$\mathcal{P}(\overline{w}) \otimes \left[ \bigotimes_{i \in \underline{n} \setminus \{k\}} A(w_i) \right] \otimes M(w_k) \longrightarrow M(w_*)$$

subject to natural equivariance, associativity and unitality conditions (cf. [BM09, Definition 1.1] for the 1-colored case).

*Remark 4.1.8.* If  $\mathcal{P}$  is an operad concentrated in arity  $\leq 1$  then  $\mathcal{P}$  is naturally isomorphic to the enveloping operad  $(\mathcal{P}_1)^{\mathcal{P}_0}$  of  $\mathcal{P}_0$  as a  $\mathcal{P}_1$ -algebra. Considering  $\mathcal{P}_1$  as an  $\mathcal{M}$ -enriched category and  $\mathcal{P}_0 : \mathcal{P}_1 \rightarrow \mathcal{M}$  as an enriched functor we may then identify  $\text{Alg}_{\mathcal{P}_{\leq 1}}$  with the coslice category  $\text{Fun}(\mathcal{P}_1, \mathcal{M})_{\mathcal{P}_0/}$ . For example, if  $A$  is a  $\mathcal{P}$ -algebra then the category of  $\mathcal{P}_{\leq 1}^A$ -algebras is equivalent to the category of  $\mathcal{P}_1^A$ -algebras under  $\mathcal{P}_0^A$ , i.e.  $A$ -modules  $M$  equipped with a map of  $A$ -modules  $A \rightarrow M$ . Similarly, the operad  $\mathcal{P}_{\leq 0}^{A,+} = (\mathcal{P}^A)_{\leq 0}^+$  is the operad whose algebras are objects  $V \in \mathcal{M}^W$  equipped with a map  $A \rightarrow V$  in  $\mathcal{M}^W$ .

## 4.2 The comparison theorem

In this section we will specialize to the case where  $\mathcal{M}$  is not just an SM locally presentable category, but a **combinatorial SM model category**. Recall that an operad  $\mathcal{P}$  is called **admissible** if the model structure on  $\mathcal{M}$  transfers to the category  $\text{Alg}_{\mathcal{P}}$  of  $\mathcal{P}$ -algebras. When  $\mathcal{P}$  is admissible we will also consider the category  $\text{Alg}_{\mathcal{P}}^{\text{aug}}$  of augmented algebras as a model category with its slice model structure. We will say that  $\mathcal{P}$  is **stably admissible** if it is admissible and in addition the stable model structure on  $\text{Sp}(\text{Alg}_{\mathcal{P}}^{\text{aug}})$  exists.

One case where stable admissibility is easy to verify is when  $\mathcal{P}$  is 1-skeletal, i.e.,  $\mathcal{P} = \mathcal{P}_{\leq 1}$ . Indeed, recall from Remark 4.1.8 that a 1-skeletal operad  $\mathcal{P}$  is simply an  $\mathcal{M}$ -enriched category  $\mathcal{P}_1$  together with an enriched functor  $\mathcal{P}_0 : \mathcal{P}_1 \rightarrow \mathcal{M}$ . The category of  $\mathcal{P}$ -algebras is then equivalent to the category  $\text{Fun}(\mathcal{P}_1, \mathcal{M})_{\mathcal{P}_0/}$  of enriched functors  $\mathcal{P}_1 \rightarrow \mathcal{M}$  under  $\mathcal{P}_0$ . In this case we can endow  $\text{Fun}(\mathcal{P}_1, \mathcal{M})_{\mathcal{P}_0/}$  with the coslice model structure associated to the projective model structure on  $\text{Fun}(\mathcal{P}_1, \mathcal{M})$ . Under the equivalence of categories  $\text{Alg}_{\mathcal{P}} \cong \text{Fun}(\mathcal{P}_1, \mathcal{M})_{\mathcal{P}_0/}$  this model structure is the one transferred from  $\mathcal{M}^W$ . In particular, any 1-skeletal operad in  $\mathcal{M}$  is admissible. Furthermore, if  $\mathcal{M}$  is left proper then  $\text{Alg}_{\mathcal{P}}^{\text{aug}} \cong \text{Fun}(\mathcal{P}_1, \mathcal{M})_{\mathcal{P}_0//\mathcal{P}_0}$  is left proper and hence  $\mathcal{P}$  is stably admissible.

Our goal in this section is to prove the core comparison results of this paper, which relate the stabilization of  $\text{Alg}_{\mathcal{P}}^{\text{aug}}$  to the stabilization of the simpler category  $\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}}$ . First recall that the map  $\varphi : \mathcal{P}_{\leq 1} \rightarrow \mathcal{P}$  induces an adjunction

$$\varphi_!^{\text{aug}} : \text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}} \xrightleftharpoons[\perp]{} \text{Alg}_{\mathcal{P}}^{\text{aug}} : \varphi_{\text{aug}}^*$$

on augmented algebras (see Remark 3.1.5) and hence an adjunction

$$\varphi_!^{\text{Sp}} := \text{Sp}(\varphi_!^{\text{aug}}) : \text{Sp}(\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}}) \xrightleftharpoons[\perp]{} \text{Sp}(\text{Alg}_{\mathcal{P}}^{\text{aug}}) : \text{Sp}(\varphi_{\text{aug}}^*) =: \varphi_{\text{Sp}}^*$$

on spectrum objects. Finally, recall that an operad  $\mathcal{P}$  is called  **$\Sigma$ -cofibrant** if the underlying



symmetric sequence of  $\mathcal{P}$  is projectively cofibrant. Our main theorem can then be formulated as follows:

**THEOREM 4.2.1.** *Let  $\mathcal{M}$  be a differentiable, left proper, combinatorial SM model category and let  $\mathcal{P}$  be a  $\Sigma$ -cofibrant stably admissible operad in  $\mathcal{M}$ . Assume either that  $\mathcal{M}$  is right proper or that  $\mathcal{P}_0$  is fibrant. Then the induced Quillen adjunction*

$$\varphi_!^{\text{Sp}} : \text{Sp}(\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}}) \xrightleftharpoons{\perp} \text{Sp}(\text{Alg}_{\mathcal{P}}^{\text{aug}}) : \varphi_{\text{Sp}}^*$$

*is a Quillen equivalence.*

*Remark 4.2.2.* If every object in  $\mathcal{M}$  is cofibrant and  $\mathcal{P}$  is a single-colored cofibrant operad then  $\mathcal{P}$  is admissible ([PS14, Theorem 1.0.2]) and the association  $A \mapsto \mathcal{P}^A$  preserves weak equivalences ([Fre09, 17.4.B(b)]). This implies that  $\text{Alg}_{\mathcal{P}}$  (as well as  $\text{Alg}_{\mathcal{P}}^{\text{aug}}$ ) is left proper and hence that  $\mathcal{P}$  is stably admissible. The work of [Rez02] gives the same conclusion for a colored cofibrant operad when  $\mathcal{M}$  is the category of simplicial sets. It seems very likely that this statement holds for every cofibrant colored operad and every combinatorial model category  $\mathcal{M}$  in which every object is cofibrant.

The key ingredient in Theorem 4.2.1 is embodied in the following proposition, which does not assume that  $\mathcal{P}$  is stably admissible.

**PROPOSITION 4.2.3.** *Let  $\mathcal{M}$  be a differentiable, left proper, combinatorial SM model category and let  $\mathcal{P}$  be a  $\Sigma$ -cofibrant operad in  $\mathcal{M}$ . Assume either that  $\mathcal{M}$  is right proper or that  $\mathcal{P}_0$  is fibrant. Consider the induced Quillen adjunction on  $\mathbb{N} \times \mathbb{N}$ -diagrams, (abusively) denoted by*

$$\varphi_!^{\text{aug}} : (\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}_{\text{inj}} \xrightleftharpoons{\perp} (\text{Alg}_{\mathcal{P}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}_{\text{inj}} : \varphi_{\text{aug}}^*.$$

*Then the following two statements hold:*

- (1) *the right derived functor  $\mathbb{R}\varphi_{\text{aug}}^*$  preserves and detects stable equivalences between pre-spectra.*
- (2) *for any levelwise cofibrant pre-spectrum object  $X_{\bullet\bullet}$  in  $\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}}$ , the derived unit map  $u_X^h : X_{\bullet\bullet} \rightarrow \mathbb{R}\varphi_{\text{aug}}^* \varphi_!^{\text{aug}} X_{\bullet\bullet}$  is a stable equivalence.*

We note that Theorem 4.2.1 is a direct consequence of Proposition 4.2.3. With an eye towards future applications, Proposition 4.2.3 was designed to give a slightly more general result, mostly in the sense that it does not require the assumption that the stable model structure on  $\text{Sp}(\text{Alg}_{\mathcal{P}}^{\text{aug}})$  exists.

To describe an analogue of Theorem 4.2.1 in this more general setting let us recall some notation from §2. For a weakly pointed combinatorial model category  $\mathcal{N}$  let us denote by  $\text{Sp}'(\mathcal{N}) \subseteq \mathcal{N}^{\mathbb{N} \times \mathbb{N}}$  the full subcategory spanned by  $\Omega$ -spectra, considered as a **relative category** with respect to levelwise equivalences, and by  $\text{Sp}''(\mathcal{N}) \subseteq \mathcal{N}^{\mathbb{N} \times \mathbb{N}}$  the full subcategory spanned by pre-spectra, considered as a relative category with respect to **stable equivalences**. We recall that the inclusion  $\text{Sp}'(\mathcal{N}) \subseteq \text{Sp}''(\mathcal{N})$  is an equivalence of relative categories and that the underlying  $\infty$ -categories of both  $\text{Sp}'(\mathcal{N})$  and  $\text{Sp}''(\mathcal{N})$  model the  $\infty$ -categorical stabilization  $\text{Sp}(\mathcal{N}_{\infty})$  (see Remarks 2.2.9 and 2.2.10).

**COROLLARY 4.2.4.** *Let  $\mathcal{M}$  be a differentiable, combinatorial SM model category and let  $\mathcal{P}$  be a  $\Sigma$ -cofibrant operad in  $\mathcal{M}$ . Assume either that  $\mathcal{M}$  is right proper or that  $\mathcal{P}_0$  is fibrant. Then the functor*

$$\text{Sp}'(\text{Alg}_{\mathcal{P}}^{\text{aug}}) \longrightarrow \text{Sp}'(\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})$$

induced by the forgetful functor is an equivalence of relative categories. In particular,  $\mathrm{Sp}(\mathrm{Alg}_{\mathcal{P}_{\leq 1}}^{\mathrm{aug}})$  is a model for the stabilization of  $(\mathrm{Alg}_{\mathcal{P}}^{\mathrm{aug}})_{\infty}$

*Proof.* Since the functor  $\mathbb{R}\varphi_{\mathrm{aug}}^* : (\mathrm{Alg}_{\mathcal{P}}^{\mathrm{aug}})^{\mathbb{N} \times \mathbb{N}} \rightarrow (\mathrm{Alg}_{\mathcal{P}_{\leq 1}}^{\mathrm{aug}})^{\mathbb{N} \times \mathbb{N}}$  preserves  $\Omega$ -spectra it follows that  $\mathbb{L}\varphi_{\mathrm{aug}}^*$  preserves stable equivalences. By Proposition 4.2.3(1)  $\mathbb{R}\varphi_{\mathrm{aug}}^*$  preserves stable equivalences between pre-spectra. It follows that  $\mathbb{L}\varphi_{\mathrm{aug}}^*$  and  $\mathbb{R}\varphi_{\mathrm{aug}}^*$  induce relative functors between  $\mathrm{Sp}''(\mathrm{Alg}_{\mathcal{P}}^{\mathrm{aug}})$  and  $\mathrm{Sp}''(\mathrm{Alg}_{\mathcal{P}_{\leq 1}}^{\mathrm{aug}})$ . Combining (1) and (2) of Proposition 4.2.3 we may conclude that the compositions  $\mathbb{R}\varphi_{\mathrm{aug}}^* \circ \mathbb{L}\varphi_{\mathrm{aug}}^*$  and  $\mathbb{L}\varphi_{\mathrm{aug}}^* \circ \mathbb{R}\varphi_{\mathrm{aug}}^*$  are both related to the corresponding identities by chains of natural weak equivalences. In particular,  $\mathbb{R}\varphi_{\mathrm{aug}}^* : \mathrm{Sp}''(\mathrm{Alg}_{\mathcal{P}}^{\mathrm{aug}}) \rightarrow \mathrm{Sp}''(\mathrm{Alg}_{\mathcal{P}_{\leq 1}}^{\mathrm{aug}})$  is an equivalence of relative categories and hence  $\mathrm{Sp}'(\mathrm{Alg}_{\mathcal{P}}^{\mathrm{aug}}) \rightarrow \mathrm{Sp}'(\mathrm{Alg}_{\mathcal{P}_{\leq 1}}^{\mathrm{aug}})$  is an equivalence of relative categories as well.  $\square$

The rest of this section is devoted to the proof of Proposition 4.2.3. We begin with some preliminary lemmas. We will say that a map  $f : X \rightarrow Y$  is **null-homotopic** if its image in  $\mathrm{Ho}(\mathcal{M})$  factors through the zero object 0.

**DEFINITION 4.2.5.** Let  $\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{N}$  be weakly pointed model categories and let  $\mathcal{F} : \prod_i \mathcal{M}_i \rightarrow \mathcal{N}$  be a functor (of ordinary categories). We will say that  $\mathcal{F}$  is **multi-reduced** if  $\mathcal{F}(X_1, \dots, X_n)$  is a weak zero object of  $\mathcal{N}$  whenever all the  $X_i$  are cofibrant and at least one of them is a weak zero object.

**LEMMA 4.2.6** (cf. [Lur14, Proposition 6.1.3.10]). *Let  $\mathcal{M}_1, \dots, \mathcal{M}_n$  and  $\mathcal{N}$  be combinatorial differentiable weakly pointed model categories and let  $\mathcal{F} : \prod_i \mathcal{M}_i \rightarrow \mathcal{N}$  be a multi-reduced functor. For every collection  $Z_{\bullet\bullet}^i \in \mathcal{M}_i^{\mathbb{N} \times \mathbb{N}}$  of levelwise cofibrant pre-spectrum objects, the object  $\mathcal{F}(Z_{\bullet\bullet}^1, \dots, Z_{\bullet\bullet}^n)$  is stably equivalent to a weak zero object.*

*Proof.* For simplicity we will prove the claim for  $n = 2$ . The proof in the general case is similar. Since  $\mathcal{F}$  is multi-reduced we have that  $\mathcal{F}(X, Z_{k,m}^i)$  and  $\mathcal{F}(Z_{m,k}^i, X)$  are weak zero objects for every  $i = 1, 2$ ,  $k \neq m$  and any cofibrant  $X \in \mathcal{M}$ . Consider the following commutative diagram

$$\begin{array}{ccccc}
 \mathcal{F}(Z_{n,n}^1, Z_{n,n}^2) & \longrightarrow & \mathcal{F}(Z_{n,n+1}^1, Z_{n,n}^2) & \xrightarrow{\simeq} & \mathcal{F}(Z_{n,n+1}^1, Z_{n,n+1}^2) \\
 \downarrow & & \downarrow & & \downarrow \simeq \\
 \mathcal{F}(Z_{n+1,n}^1, Z_{n,n}^2) & \longrightarrow & \mathcal{F}(Z_{n+1,n+1}^1, Z_{n,n}^2) & \longrightarrow & \mathcal{F}(Z_{n+1,n+1}^1, Z_{n,n+1}^2) \\
 \downarrow \simeq & & \downarrow & & \downarrow \\
 \mathcal{F}(Z_{n+1,n}^1, Z_{n+1,n}^2) & \xrightarrow{\simeq} & \mathcal{F}(Z_{n+1,n+1}^1, Z_{n+1,n}^2) & \longrightarrow & \mathcal{F}(Z_{n+1,n+1}^1, Z_{n+1,n+1}^2)
 \end{array}$$

We first note that all off-diagonal items in this diagram are weak zero objects. The external square induces a map  $f_n : \Sigma \mathcal{F}(Z_{n,n}^1, Z_{n,n}^2) \rightarrow \mathcal{F}(Z_{n+1,n+1}^1, Z_{n+1,n+1}^2)$  in the homotopy category  $\mathrm{Ho}(\mathcal{N})$ , which factors as

$$\Sigma \mathcal{F}(Z_{n,n}^1, Z_{n,n}^2) \rightarrow \mathcal{F}(Z_{n+1,n+1}^1, Z_{n,n}^2) \rightarrow \mathcal{F}(Z_{n+1,n+1}^1, Z_{n+1,n+1}^2)$$

where the first map is induced from the top left square. Since the second map is null-homotopic, it follows that the map  $f_n$  is null-homotopic as well. By Corollary 2.4.6  $\mathcal{F}(Z_{\bullet\bullet}^1, Z_{\bullet\bullet}^2)$  is stably equivalent to an  $\Omega$ -spectrum whose value at the place  $(m, m)$  can be computed as a homotopy colimit of the form

$$\mathcal{F}(Z_{m,m}^1, Z_{m,m}^2) \xrightarrow{g_m} \Omega \mathcal{F}(Z_{m+1,m+1}^1, Z_{m+1,m+1}^2) \xrightarrow{g_{m+1}} \Omega^2 \mathcal{F}(Z_{m+2,m+2}^1, Z_{m+2,m+2}^2) \rightarrow \dots$$

where the image of  $g_i$  in  $\text{Ho}(\mathcal{N})$  is adjoint to  $f_i$  and hence null-homotopic for every  $i \geq m$ . Since a homotopy colimit of a sequence of null-homotopic maps is a weak zero object the desired result follows.  $\square$

Recall that for any map  $f : X \rightarrow Y$  in a category with a zero object  $0$ , the **cofiber** of  $f$ , denoted  $\text{cof}(f)$ , is the object sitting in the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{cof}(f) \end{array} \quad (4.2.1)$$

LEMMA 4.2.7. *Let  $\mathcal{M}$  be a strictly pointed combinatorial model category and suppose that  $f : X \rightarrow Y$  is a levelwise cofibration between levelwise cofibrant pre-spectra in  $\mathcal{M}$ . Then  $f$  is a stable equivalence if and only if the map  $0 \rightarrow \text{cof}(f)$  is a stable equivalence.*

*Proof.* Under the assumptions of lemma the square 4.2.1 is homotopy coCartesian in  $\mathcal{M}^{\mathbb{N} \times \mathbb{N}}$ . It follows that if  $f$  is a stable equivalence then  $0 \rightarrow \text{cof}(f)$  is a stable equivalence. We shall now show that if  $0 \rightarrow \text{cof}(f)$  is a stable equivalence then  $f$  is a stable equivalence. Note that if the model structure on  $\text{Sp}(\mathcal{M})$  exists then this is simply a consequence of Corollary 2.2.6 which implies that 4.2.1 becomes homotopy Cartesian when considered in  $\text{Sp}(\mathcal{M})$ . If the stable model structure on  $\text{Sp}(\mathcal{M})$  does not exist one can see this formally by extending 4.2.1 to a diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z_1 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{cof}(f) & \longrightarrow & X' \\ & & \downarrow & & \downarrow \\ & & Z_2 & \longrightarrow & Y' \end{array}$$

in which all the squares are homotopy coCartesian and  $Z_1, Z_2$  are weak zero objects. If  $0 \rightarrow \text{cof}(f)$  is a stable equivalence then the map  $\text{cof}(f) \rightarrow Z_2$  is a stable equivalence and hence the map  $X' \rightarrow Y'$  is a stable equivalence. On the other hand, since the external rectangles are homotopy coCartesian it follows that the map  $X' \rightarrow Y'$  is a model for the induced map  $\Sigma X \rightarrow \Sigma Y$  on suspensions. Now for every  $\Omega$ -spectra  $W$  we have  $\text{Map}^h(X', W[1]) \simeq \text{Map}^h(X, \Omega W[1]) \simeq \text{Map}^h(X, W)$  and the same for  $Y'$ . Since  $X' \rightarrow Y'$  is a stable equivalence it now follows that  $f : X \rightarrow Y$  is a stable equivalence.  $\square$

COROLLARY 4.2.8. *Let  $\mathcal{M}$  be a combinatorial differentiable SM model category and let  $A^1, \dots, A^n \in \mathcal{M}$  be a collection of cofibrant objects (with  $n \geq 2$ ). For each  $i = 1, \dots, n$  let  $A^i \xrightarrow{f_{\bullet\bullet}^i} X_{\bullet\bullet}^i \rightarrow A^i$  be a levelwise cofibrant pre-spectrum object in  $\mathcal{M}_{A^i//A^i}$ . Then the levelwise pushout-product*

$$f_{\bullet\bullet}^1 \square \dots \square f_{\bullet\bullet}^n : Q(f_{\bullet\bullet}^1, \dots, f_{\bullet\bullet}^n) \rightarrow X_{\bullet\bullet}^{\otimes n}$$

*is a stable equivalence and levelwise cofibration between levelwise cofibrant pre-spectrum objects in  $\mathcal{M}_{A^1 \otimes \dots \otimes A^n // A^1 \otimes \dots \otimes A^n}$ .*

*Proof.* The pushout-product axiom in  $\mathcal{M}$  implies that  $f_{\bullet\bullet}^1 \square \dots \square f_{\bullet\bullet}^n$  is a levelwise cofibration between levelwise cofibrant objects. By Lemma 4.2.7 it will now suffice to show that the cofiber of this map is stably equivalent to the zero pre-spectrum in  $\mathcal{M}_{A^1 \otimes \dots \otimes A^n // A^1 \otimes \dots \otimes A^n}$ .

Consider the functor  $\mathcal{G} : \prod_{i=1}^n \mathcal{M}_{A^i//A^i} \longrightarrow \mathcal{M}_{A^1 \otimes \dots \otimes A^n // A^1 \otimes \dots \otimes A^n}$  given by

$$\mathcal{G}(A^1 \xrightarrow{f^1} X^1 \longrightarrow A^1, \dots, A^n \xrightarrow{f^n} X^n \longrightarrow A^n) = \text{cof}(f^1 \square \dots \square f^n).$$

This functor is multi-reduced: indeed, the cofiber  $\text{cof}(f^1 \square \dots \square f^n)$  is a levelwise weak zero object if at least one of the  $f^i$  is a trivial cofibration in  $\mathcal{M}$ , by the pushout-product axiom. Lemma 4.2.6 now implies that the cofiber of the map  $f_{\bullet\bullet}^1 \square \dots \square f_{\bullet\bullet}^n$  is stably equivalent to a zero object, as desired.  $\square$

Let us now fix a combinatorial SM model category  $\mathcal{M}$ , a set of colors  $W$  and a  $W$ -colored operad  $\mathcal{P}$  in  $\mathcal{M}$ . We will be interested in the following maps of operads

$$\begin{array}{ccc} \mathcal{O} = \mathcal{P}_{\leq 0}^+ & \xrightarrow{\rho} & \mathcal{P} \\ & \searrow \psi & \nearrow \varphi \\ & \mathcal{P}_{\leq 1} & \end{array} \quad (4.2.2)$$

where  $\varphi$  is the natural ‘inclusion’ and  $\psi$  is induced from the map of symmetric sequences  $\mathcal{P}_{\leq 0} \longrightarrow \mathcal{P}_{\leq 1}$ . Upon passing to operadic algebras, this sequence yields a sequence of adjunctions:

$$\text{Alg}_{\mathcal{O}} \begin{array}{c} \xrightarrow{\psi_!} \\ \perp \\ \xleftarrow{\psi^*} \end{array} \text{Alg}_{\mathcal{P}_{\leq 1}} \begin{array}{c} \xrightarrow{\varphi_!} \\ \perp \\ \xleftarrow{\varphi^*} \end{array} \text{Alg}_{\mathcal{P}} \quad (4.2.3)$$

By Remark 4.1.8 we may identify  $\mathcal{P}_{\leq 1}$ -algebras with enriched functors  $\mathcal{P}_1 \longrightarrow \mathcal{M}$  under  $\mathcal{P}_0$  and  $\mathcal{O}$ -algebras with objects in  $\mathcal{M}^W$  under  $\mathcal{P}_0$ . Now if  $X$  is an  $\mathcal{O}$ -algebra then the map of  $\mathcal{P}_{\leq 1}$ -algebras  $\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X \longrightarrow \mathcal{P} \circ_{\mathcal{O}} X$  can be factored as a transfinite composition (see §A)

$$\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X \longrightarrow \mathcal{P}_{\leq 2} \circ_{\mathcal{O}} X \longrightarrow \dots \longrightarrow \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X \longrightarrow \dots \quad (4.2.4)$$

such that for every  $n \geq 2$  we have a pushout square  $\mathcal{P}_{\leq 1}$ -algebras of the form

$$\begin{array}{ccc} R_n^-(X) & \longrightarrow & R_n^+(X) \\ \downarrow & & \downarrow \\ \mathcal{P}_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X \end{array} \quad (4.2.5)$$

where  $R_n^-(X)$  and  $R_n^+(X)$  are described in §A. Now let  $X$  be an **augmented**  $\mathcal{O}$ -algebra, so that  $X$  is equipped with a map  $X \longrightarrow \mathcal{O}_0 \cong \mathcal{P}_0$  to the initial  $\mathcal{O}$ -algebra. We note that the free functor  $\rho_! : \text{Alg}_{\mathcal{O}} \longrightarrow \text{Alg}_{\mathcal{P}}$  is a left adjoint and hence preserves initial objects and augmented objects. In particular,  $\rho_!(X) = \mathcal{P} \circ_{\mathcal{O}} X$  carries a natural map to the initial  $\mathcal{P}$ -algebra  $\mathcal{P} \circ_{\mathcal{O}} \mathcal{O}_0 \cong \mathcal{P}_0$ . We note that  $\mathcal{P}_0$  is also initial as a  $\mathcal{P}_{\leq 1}$ -algebra, and hence the augmentation of  $\mathcal{P} \circ_{\mathcal{O}} X$  induces (by composition) an augmentation on each  $\mathcal{P}_{\leq n} \circ_{\mathcal{O}} X$  and on each  $R_n^-(X), R_n^+(X)$ . As a result, we may (and will) naturally consider 4.2.4 to be a filtration of  $\varphi_{\text{aug}}^* \rho_!^{\text{aug}}(X) = \mathcal{P} \circ_{\mathcal{O}} X$  as an augmented  $\mathcal{P}_{\leq 1}$ -algebra and the squares 4.2.5 to be pushout squares of augmented  $\mathcal{P}_{\leq 1}$ -algebras.

**PROPOSITION 4.2.9.** *Let  $X_{\bullet\bullet} \in (\text{Alg}_{\mathcal{O}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$  be a levelwise cofibrant pre-spectrum object in augmented  $\mathcal{O}$ -algebras. If  $\mathcal{P}$  is  $\Sigma$ -cofibrant then the induced map*

$$u : \mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet} \longrightarrow \mathcal{P} \circ_{\mathcal{O}} X_{\bullet\bullet}$$

*is a stable weak equivalence in  $(\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$ .*

*Proof.* Recall that the initial  $\mathcal{O}$ -algebra  $\mathcal{O}_0$  is the free  $\mathcal{O}$ -algebra  $\mathcal{O} \circ \emptyset$  on the initial object in  $\mathcal{M}$ . It follows that  $\mathcal{P}_{\leq n} \circ_{\mathcal{O}} \mathcal{O}_0 = \mathcal{P}_{\leq n} \circ_{\mathcal{O}} (\mathcal{O} \circ \emptyset) = \mathcal{P}_0$  so that each  $\mathcal{P}_{\leq n} \circ_{\mathcal{O}} X_{\bullet\bullet}$  is a pre-spectrum in  $\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}}$ .

Furthermore, since each  $X_{m,k}$  is levelwise cofibrant as an  $\mathcal{O}$ -algebra we get that  $\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{m,k}$  is levelwise cofibrant as an  $\mathcal{P}_{\leq 1}$ -algebra. Identifying  $\mathcal{P}_{\leq 1}$ -algebras with functors  $\mathcal{P}_1 \rightarrow \mathcal{M}$  under  $\mathcal{P}_0$  (and the transferred model structure with the cosliced projective one) we may conclude that the underlying  $\mathcal{O}$ -algebra of  $\mathcal{P}_1 \circ_{\mathcal{O}} X_{m,k}$  is cofibrant as well.

Now the map  $u$  is a transfinite composition of the maps

$$\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet} \xrightarrow{u_2} \mathcal{P}_{\leq 2} \circ_{\mathcal{O}} X_{\bullet\bullet} \xrightarrow{u_3} \dots \xrightarrow{u_n} \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X_{\bullet\bullet} \rightarrow \dots$$

Let  $\psi_{\text{aug}}^* : (\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}} \rightarrow (\text{Alg}_{\mathcal{O}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$  be the induced right Quillen functor. By Proposition 4.2.10 the functor  $\psi_{\text{aug}}^*$  is differentiable. Since  $\psi_{\text{aug}}^*$  preserves and detects weak equivalences, the second part of Corollary 2.4.8 implies that  $\psi_{\text{aug}}^*$  detects weak equivalences between pre-spectra. By Remark 2.1.9 and since  $\psi_{\text{aug}}^*(\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet})$  is levelwise cofibrant it will hence suffice to prove that the maps

$$\psi_{\text{aug}}^*(\mathcal{P}_{\leq n-1} \circ_{\mathcal{O}} X_{\bullet\bullet}) \rightarrow \psi_{\text{aug}}^*(\mathcal{P}_{\leq n} \circ_{\mathcal{O}} X_{\bullet\bullet})$$

are stable equivalences and levelwise cofibrations of pre-spectra  $\text{Alg}_{\mathcal{O}}^{\text{aug}}$ .

We note that the functor  $\psi^* : \text{Alg}_{\mathcal{P}_{\leq 1}} \rightarrow \text{Alg}_{\mathcal{O}}$  is simply the induced restriction  $\mathcal{M}_{\mathcal{P}_0/\mathcal{P}_1}^{\mathcal{P}_1} \rightarrow \mathcal{M}_{\mathcal{P}_0/\mathcal{P}_0}^W$  and hence admits both a left and a right adjoint. It follows that  $\psi_{\text{aug}}^*$  preserves colimits and in particular the pushout square 4.2.5. By Remark 2.1.9 it will now suffice to show that for every  $w_0 \in W$  the map

$$\psi_{\text{aug}}^*(R_n^-(X_{\bullet\bullet}))(w_0) \rightarrow \psi_{\text{aug}}^*(R_n^+(X_{\bullet\bullet}))(w_0) \quad (4.2.6)$$

is a stable equivalence and a levelwise cofibration between levelwise cofibrant pre-spectrum objects in  $\mathcal{M}_{\mathcal{P}_0(w_0)/\mathcal{P}_0(w_0)}$ .

Let us now fix a color  $w_0 \in W$  and a number  $n \geq 1$ . Let  $\Sigma_{w_0}^n \subseteq \Sigma_W^n$  the full subgroupoid spanned by those  $\bar{w} \in \Sigma_W^n$  such that  $w_* = w_0$  (see §4.1, §A). We have the injectively cofibrant functor  $\mathcal{P}_0^{\otimes n} : \Sigma_{w_0}^n \rightarrow \mathcal{M}$  given by  $\mathcal{P}_0^{\otimes n}(\bar{w}) = \bigotimes_{i \in \underline{n}} \mathcal{P}_0(w_i)$ . For  $k, m \in \mathbb{N}$  consider the functors  $X_{k,m}^{\otimes n}, Q_{k,m} : \Sigma_{w_0}^n \rightarrow \mathcal{M}$  with  $X_{k,m}^{\otimes n}(\bar{w}) = \bigotimes_{i \in \underline{n}} X_{k,m}(w_i)$  and such that  $Q_{k,m}(\bar{w})$  is the codomain of the pushout-product of the maps  $\mathcal{P}_0(w_i) \rightarrow X_{k,m}(w_i)$  for  $i = 1, \dots, n$ . Corollary 4.2.8 now implies that the natural map

$$Q(X_{\bullet\bullet}) \rightarrow X_{\bullet\bullet}^{\otimes n} \quad (4.2.7)$$

is a stable equivalence and a levelwise cofibration between levelwise cofibrant pre-spectrum objects in  $(\mathcal{M}_{\text{inj}}^{\Sigma_{w_0}^n})_{\mathcal{P}_0^{\otimes n}/\mathcal{P}_0^{\otimes n}}$ .

Now recall that the coend operation  $\mathcal{M}_{\text{proj}}^{\Sigma_{w_0}^n} \times \mathcal{M}_{\text{inj}}^{\Sigma_{w_0}^n} \rightarrow \mathcal{M}$ , which we will denote by  $F, G \mapsto F \otimes_{\Sigma_{w_0}^n} G$ , is a left Quillen bifunctor (see, e.g., [Lur09, Remark A.2.9.27]). Let  $\mathcal{P}_{w_0}^n : \Sigma_{w_0}^n \rightarrow \mathcal{M}$  be the functor  $\bar{w} \mapsto \mathcal{P}(\bar{w})$ . Since  $\mathcal{P}$  is  $\Sigma$ -cofibrant we have that  $\mathcal{P}_{w_0}^n$  is projectively cofibrant, and so we may consider the left Quillen functor  $\mathcal{L}$  given by the composition

$$\mathcal{L} : (\mathcal{M}_{\text{inj}}^{\Sigma_{w_0}^n})_{\mathcal{P}_0^{\otimes n}/\mathcal{P}_0^{\otimes n}} \xrightarrow{\mathcal{P}_{w_0}^n \otimes_{\Sigma_{w_0}^n} (-)} \mathcal{M}_{\mathcal{P}_{w_0}^n \otimes_{\Sigma_{w_0}^n} \mathcal{P}_0^{\otimes n}/\mathcal{P}_0^{\otimes n}} \rightarrow \mathcal{M}_{\mathcal{P}_0(w_0)/\mathcal{P}_0(w_0)}$$

where the second functor is the cobase change along the map  $\mathcal{P}_{w_0}^n \otimes_{\Sigma_{w_0}^n} \mathcal{P}_0^{\otimes n} \rightarrow \mathcal{P}_0(w_0)$  induced by the  $\mathcal{P}$ -algebra structure of  $\mathcal{P}_0$ . Corollary A.0.4 now tells us that the map (4.2.6) is obtained by levelwise applying (the augmented version of)  $\mathcal{L}$  to the map (4.2.7), and is hence a stable equivalence and a levelwise cofibration between levelwise cofibrant pre-spectra, as desired.  $\square$

We are now almost ready to prove Proposition 4.2.3. Before that, let us quickly recall the following result, which is essentially contained in [PS14]:

PROPOSITION 4.2.10. *Let  $\mathcal{M}$  be a differentiable SM model category and let  $f : \mathcal{P} \rightarrow \mathcal{Q}$  be a map of  $\Sigma$ -cofibrant admissible operads in  $\mathcal{M}$ .*

*Then the derived forgetful functor  $\mathbb{R}f_{\text{aug}}^* : \text{Alg}_{\mathcal{Q}}^{\text{aug}} \rightarrow \text{Alg}_{\mathcal{P}}^{\text{aug}}$  preserves and detects weak equivalences and preserves and detects sifted homotopy colimits. Furthermore, the functor  $f_{\text{aug}}^*$  is differentiable and the induced adjunction of  $\infty$ -categories*

$$(f_{\text{aug}}^*)_{\infty} : (\text{Alg}_{\mathcal{P}}^{\text{aug}})_{\infty} \xrightleftharpoons{\perp} (\text{Alg}_{\mathcal{Q}}^{\text{aug}})_{\infty} : (f_{\text{aug}}^*)_{\infty} \quad (4.2.8)$$

*is monadic.*

*Proof.* Since the model structures on both  $\text{Alg}_{\mathcal{P}}$  and  $\text{Alg}_{\mathcal{Q}}$  are transferred from  $\mathcal{M}$  we see that  $f_{\text{aug}}^*$  preserves and detects weak equivalences. By [PS14, Proposition 7.8] both derived forgetful functors  $\text{Alg}_{\mathcal{P}} \rightarrow \mathcal{M}^W$  and  $\text{Alg}_{\mathcal{Q}} \rightarrow \mathcal{M}^W$  preserve and detect sifted homotopy colimits. Since the forgetful functors  $\text{Alg}_{\mathcal{P}}^{\text{aug}} \rightarrow \text{Alg}_{\mathcal{P}}$  and  $\text{Alg}_{\mathcal{Q}}^{\text{aug}} \rightarrow \text{Alg}_{\mathcal{Q}}$  preserve and detect homotopy colimits indexed by weakly contractible categories we may now conclude that  $f_{\text{aug}}^*$  preserves and detects sifted homotopy colimits. Furthermore, since sequential diagrams are in particular sifted and  $\mathcal{M}$  is differentiable we now get that  $\text{Alg}_{\mathcal{P}}^{\text{aug}}, \text{Alg}_{\mathcal{Q}}^{\text{aug}}$  are differentiable and that  $f_{\text{aug}}^* \dashv f_{\text{aug}}$  is a differentiable Quillen adjunction. The last claim is just an application of the  $\infty$ -categorical Barr-Beck theorem (see [Lur14, Theorem 4.7.4.5]).  $\square$

*Proof of Proposition 4.2.3.* By 4.2.10 the right Quillen functor  $\varphi^* : \text{Alg}_{\mathcal{P}}^{\text{aug}} \rightarrow \text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}}$  is differentiable and hence by Corollary 2.4.8 the right derived functor  $\mathbb{R}\varphi_{\text{aug}}^* : (\text{Alg}_{\mathcal{P}}^{\text{aug}})_{\text{inj}}^{\mathbb{N} \times \mathbb{N}} \rightarrow (\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})_{\text{inj}}^{\mathbb{N} \times \mathbb{N}}$  preserves stable weak equivalences between pre-spectra. This proves Claim (1). Let us now prove (2).

We first note that by Remark 2.3.4 every  $(\mathbb{N} \times \mathbb{N})$ -diagram is stably equivalent to a pre-spectrum object. Furthermore, the collection of pre-spectra and the collection of stable weak equivalences are both closed under homotopy colimits of  $(\mathbb{N} \times \mathbb{N})$ -diagrams. By Proposition 4.2.10 we have that  $\mathbb{R}\varphi_{\text{aug}}^*$  preserves sifted homotopy colimits (since these are computed levelwise).

This means that the collection of levelwise cofibrant pre-spectra  $X_{\bullet\bullet} \in (\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$  for which the derived unit

$$u_X^h : X_{\bullet\bullet} \rightarrow \mathbb{R}\varphi_{\text{aug}}^* \varphi_{\text{aug}}^{\text{aug}} X_{\bullet\bullet}$$

is a stable weak equivalence is closed under sifted homotopy colimits in  $(\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$ . Since homotopy colimits and weak equivalences in functor categories are computed levelwise Proposition 4.2.10 implies that the free-forgetful adjunction

$$(\text{Alg}_{\mathcal{O}}^{\text{aug}})_{\infty}^{\mathbb{N} \times \mathbb{N}} \xrightleftharpoons{\perp} (\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})_{\infty}^{\mathbb{N} \times \mathbb{N}} \quad (4.2.9)$$

is a monadic adjunction of  $\infty$ -categories. We note that both functors in this adjunction preserves pre-spectrum objects. Since the collection of pre-spectrum objects is closed under homotopy colimits it follows that 4.2.9 induces a monadic adjunction on the corresponding full subcategories spanned by pre-spectra. This means that every pre-spectrum object of  $(\text{Alg}_{\mathcal{P}_{\leq 1}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$  can be written as a sifted homotopy colimit of pre-spectra of the form  $\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet}$  for  $X_{\bullet\bullet} \in (\text{Alg}_{\mathcal{O}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$  a pre-spectrum object. It will hence suffice to show that  $u_{\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet}}^h$  is a stable weak equivalence for every cofibrant pre-spectrum  $X_{\bullet\bullet} \in (\text{Alg}_{\mathcal{O}}^{\text{aug}})^{\mathbb{N} \times \mathbb{N}}$ . Since  $\varphi_{\mathbb{N} \times \mathbb{N}}^*$  preserves weak equivalences it will

suffice to prove that the actual unit map

$$u_{\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet}} : \mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet} \longrightarrow \varphi_{\mathrm{Sp}}^* \varphi_!^{\mathrm{Sp}}(\mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet}) = \mathcal{P} \circ_{\mathcal{P}_{\leq 1}} \mathcal{P}_{\leq 1} \circ_{\mathcal{O}} X_{\bullet\bullet} = \mathcal{P} \circ_{\mathcal{O}} X_{\bullet\bullet}$$

is a stable weak equivalence. But this is exactly the content of Proposition 4.2.9, and so the proof is complete.  $\square$

### 4.3 Tangent categories of algebras and modules

Let  $\mathcal{M}$  be a left proper combinatorial differentiable SM model category and let  $\mathcal{P}$  be an admissible operad in  $\mathcal{M}$ . Let  $A \in \mathcal{M}$  be a fibrant algebra such that the **tangent model structure**  $\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}} = \mathrm{Sp}((\mathrm{Alg}_{\mathcal{P}})_{A//A})$  exists. Our goal in this section is to explain how Theorem 4.2.1 can be used to identify  $\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}}$  with the stabilization of a suitable module category, or alternatively, as a suitable category of enriched lifts. If the stable model structure on  $\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}}$  does not exist one may still consider the relative category  $\mathcal{T}'_A \mathrm{Alg}_{\mathcal{P}} = \mathrm{Sp}'((\mathrm{Alg}_{\mathcal{P}})_{A//A})$  (see Remark 2.2.9) which is always a model for the associated tangent  $\infty$ -category. In this case the discussion below can be applied instead to  $\mathcal{T}'_A \mathrm{Alg}_{\mathcal{P}}$ .

Recall the enveloping operad  $\mathcal{P}^A = \mathrm{Env}(\mathcal{P}, A)$  (see §4.1) whose characteristic property is a natural equivalence of categories  $\mathrm{Alg}_{\mathcal{P}^A} \cong (\mathrm{Alg}_{\mathcal{P}})_{A/}$ . Under this equivalence, the identity map  $A \longrightarrow A$  exhibits  $A$  as the initial  $\mathcal{P}^A$ -algebra, so that  $\mathrm{Alg}_{\mathcal{P}^A}^{\mathrm{aug}} \simeq (\mathrm{Alg}_{\mathcal{P}})_{A//A}$ . We may hence write the tangent model category at  $A$  as  $\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}} = \mathrm{Sp}(\mathrm{Alg}_{\mathcal{P}^A}^{\mathrm{aug}})$ .

Theorem 4.2.1 now gives a Quillen equivalence  $\mathcal{T}_A \mathrm{Alg}_{\mathcal{P}} \simeq \mathrm{Sp}(\mathrm{Alg}_{\mathcal{P}_{\leq 1}^A}^{\mathrm{aug}})$ . The category  $\mathrm{Alg}_{\mathcal{P}_{\leq 1}^A}$  is just the category  $(\mathrm{Mod}_A^{\mathcal{P}})_{A/}$  of  $A$ -modules in  $\mathcal{M}$  under  $A$  (see Remark 4.1.8). We hence obtain the following corollary:

**COROLLARY 4.3.1.** *Let  $\mathcal{M}$  be a differentiable, left proper, combinatorial SM model category and let  $\mathcal{P}$  be an operad. Let  $A$  be a  $\mathcal{P}$ -algebra such that  $\mathcal{P}^A$  is a stably admissible and  $\Sigma$ -cofibrant operad, and assume either that  $A$  is fibrant or that  $\mathcal{M}$  is right proper. Then restriction along  $\varphi : \mathcal{P}_{\leq 1}^A \longrightarrow \mathcal{P}^A$  yields a right Quillen equivalence*

$$\varphi_{\mathrm{Sp}}^* : \mathcal{T}_A \mathrm{Alg}_{\mathcal{P}} = \mathrm{Sp}((\mathrm{Alg}_{\mathcal{P}})_{A//A}) \xrightarrow{\simeq} \mathrm{Sp}((\mathrm{Mod}_A^{\mathcal{P}})_{A//A}) = \mathcal{T}_A \mathrm{Mod}_A^{\mathcal{P}}. \quad (4.3.1)$$

Removing the conditions that  $\mathcal{M}$  is left proper and that  $\mathcal{P}_A$  is stably admissible and replacing Theorem 4.2.1 by Corollary 4.2.4 we obtain the following variant of Corollary 4.2.4:

**COROLLARY 4.3.2.** *Let  $\mathcal{M}$  be a differentiable combinatorial SM model category and let  $\mathcal{P}$  be an operad. Let  $A$  be a  $\mathcal{P}$ -algebra such that  $\mathcal{P}^A$  is  $\Sigma$ -cofibrant operad, and assume either that  $A$  is fibrant or that  $\mathcal{M}$  is right proper. Then restriction along  $\varphi : \mathcal{P}_{\leq 1}^A \longrightarrow \mathcal{P}^A$  induces an equivalence of relative categories*

$$\varphi_{\mathrm{Sp}}^* : \mathcal{T}'_A \mathrm{Alg}_{\mathcal{P}} = \mathrm{Sp}'((\mathrm{Alg}_{\mathcal{P}})_{A//A}) \xrightarrow{\simeq} \mathrm{Sp}'((\mathrm{Mod}_A^{\mathcal{P}})_{A//A}) = \mathcal{T}'_A \mathrm{Mod}_A^{\mathcal{P}}. \quad (4.3.2)$$

*Remark 4.3.3.* Work of Fresse ([Fre09]) shows that when every object in  $\mathcal{M}$  is cofibrant and  $\mathcal{P}$  is a cofibrant single colored operad then the enveloping operad  $\mathcal{P}^A$  is stably admissible and  $\Sigma$ -cofibrant for every  $\mathcal{P}$ -algebra  $A$  (see also Remark 4.2.2). This is also true when  $\mathcal{P}$  is a colored cofibrant operad and  $\mathcal{M}$  is the category of simplicial sets by work of Rezk ([Rez02]). In a different direction, if we assume that  $A$  is a cofibrant algebra, then  $\mathcal{P}^A$  is  $\Sigma$ -cofibrant as soon as  $\mathcal{P}$  is  $\Sigma$ -cofibrant (see [BM09, Proposition 2.3]). However, in this case one does not expect  $\mathcal{P}^A$  to be stably admissible in general.

To further simplify Corollary 4.3.1 we may use Remark 3.4.2 to rewrite the right hand side of (4.3.2) as the full subcategory  $\text{Fun}_{/\mathcal{M}}^{\mathcal{M}}(\mathcal{P}_1^A, \mathcal{T}\mathcal{M}) \subseteq \text{Fun}^{\mathcal{M}}(\mathcal{P}_1^A, \mathcal{T}\mathcal{M})$  consisting of those enriched functors  $\mathcal{F} : \mathcal{P}_1^A \rightarrow \mathcal{T}\mathcal{M}$  which lie above the functor  $\mathcal{P}_0^A : \mathcal{P}_1^A \rightarrow \mathcal{M}$  (corresponding to the underlying  $A$ -module of  $A$ ). We may hence rewrite Corollary 4.3.1 as follows, identifying the tangent category at  $A$  with a suitable category of  $\mathcal{M}$ -enriched lifts:

**COROLLARY 4.3.4.** *Let  $\mathcal{M}, \mathcal{P}$  and  $A$  be as in Corollary 4.3.1. Then we have a natural right Quillen equivalence*

$$\mathcal{T}_A \text{Alg}_{\mathcal{P}} \xrightarrow{\simeq} \text{Fun}_{/\mathcal{M}}^{\mathcal{M}}(\mathcal{P}_1^A, \mathcal{T}\mathcal{M}).$$

*Remark 4.3.5.* When  $\mathcal{M}$  is stable and strictly pointed the situation simplifies. Indeed, in this case  $\text{Mod}_A^{\mathcal{P}}(\mathcal{M}) \cong \text{Fun}^{\mathcal{M}}(\mathcal{P}_1^A, \mathcal{M})$  is stable, so that Lemma 3.4.3 yields a right Quillen equivalence  $\ker : \text{Mod}_A^{\mathcal{P}}(\mathcal{M})_{A//A} \xrightarrow{\simeq} \text{Mod}_A^{\mathcal{P}}(\mathcal{M})$  and hence a right Quillen equivalence  $\mathcal{T}_A \text{Mod}_A^{\mathcal{P}} \xrightarrow{\simeq} \text{Mod}_A^{\mathcal{P}}$ .

**COROLLARY 4.3.6.** *Let  $\mathcal{M}, \mathcal{P}$  and  $A$  be as in Corollary 4.3.1 and assume in addition that  $\mathcal{M}$  is stable and strictly pointed. Let  $\mathcal{K} : \text{Alg}_{A//A}^{\mathcal{P}} \rightarrow (\text{Mod}_A^{\mathcal{P}})_{A//A} \xrightarrow{\ker} \text{Mod}_A^{\mathcal{P}}$  be the composition of the forgetful functor and the kernel functor appearing in Remark 4.3.5. Then the functors*

$$\mathcal{T}_A \text{Alg}_{\mathcal{P}} \xrightarrow[\simeq]{\mathcal{K}_{\text{Sp}}} \text{Sp}(\text{Mod}_A^{\mathcal{P}}) \xrightarrow[\simeq]{\Omega^\infty} \text{Mod}_A^{\mathcal{P}}$$

*are right Quillen equivalence.*

#### 4.4 The $\infty$ -categorical comparison

Our goal in this section is to formulate and prove an  $\infty$ -categorical counterpart of Corollary 4.3.1. For this it will be useful to consider another approach for the theory of modules, where one considers the collection of pairs  $(A, M)$  of a  $\mathcal{P}$ -algebra  $A$  and an  $A$ -module  $M$  as algebras over another operad  $\mathcal{M}\mathcal{P}$ . We shall henceforth follow the approach of [Hin15]. Let  $\text{Com}$  be the commutative operad and let  $\mathcal{M}\text{Com}$  be the operad (in sets) with two colors  $W = \{a, m\}$  and such that the set of operations  $(w_1, \dots, w_n) \mapsto w_0$  is either a singleton, if  $w_0 = m$  and exactly one of the  $w_i$ 's is  $m$  or if  $w_0 = a$  and all the  $w_i$ 's are  $a$ , and empty otherwise. There are natural maps  $\text{Com} \rightarrow \mathcal{M}\text{Com} \rightarrow \text{Com}$  where the first one sends the only object of  $\text{Com}$  to  $a$  and the second is the terminal map. One can then easily verify that the data of an  $\mathcal{M}\text{Com}$ -algebra in a symmetric monoidal category  $\mathcal{C}$  is the same as a pair  $(A, M)$  where  $A$  is a commutative algebra in  $\mathcal{C}$  and  $M$  is an  $A$ -module. Restriction along the map  $\text{Com} \rightarrow \mathcal{M}\text{Com}$  induces the projection  $(A, M) \mapsto A$ .

Given a simplicial operad  $\mathcal{P}$  we will denote by

$$\mathcal{M}\mathcal{P} = \mathcal{M}\text{Com} \times_{\text{Com}} \mathcal{P}$$

the associated fiber product in the category of simplicial operads. If  $\mathcal{C}$  is a simplicial model category then one can easily verify that the data of an  $\mathcal{M}\mathcal{P}$ -algebra in  $\mathcal{C}$  is the same as a pair  $(A, M)$  where  $A$  is a  $\mathcal{P}$ -algebra in  $\mathcal{C}$  and  $M$  is an  $A$ -module.

We will denote by  $\mathcal{M}\text{Com}^\otimes = \mathcal{N}^\otimes(\mathcal{M}\text{Com})$  the operadic nerve of  $\mathcal{M}\text{Com}$ . Given an  $\infty$ -operad  $\mathcal{O}^\otimes$  we will denote by

$$\mathcal{M}\mathcal{O}^\otimes = \mathcal{M}\text{Com}^\otimes \times_{\text{Com}^\otimes} \mathcal{O}^\otimes$$

the associated (homotopy) fiber product in the model category of pre-operads. Since the operadic nerve preserves fiber products we have that if  $\mathcal{P}$  is a simplicial operad then  $\mathcal{N}^\otimes(\mathcal{M}\mathcal{P}) \cong \mathcal{M}\mathcal{N}^\otimes(\mathcal{P})$ .

**DEFINITION 4.4.1** [Hin15, Def. 5.2.1]. Let  $\mathcal{O}^\otimes$  be an  $\infty$ -operad and  $\mathcal{C}^\otimes$  a symmetric monoidal  $\infty$ -category. Let  $A \in \text{Alg}_0(\mathcal{C})$  be an  $\mathcal{O}$ -algebra object in  $\mathcal{C}$ . The  $\infty$ -category  $\text{Mod}_A^{\mathcal{O}}(\mathcal{C})$  is defined



as the fiber product

$$\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{M}\mathcal{O}}(\mathcal{C}) \times_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})} \{A\}$$

We will refer to  $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$  as the  $\infty$ -category of  $A$ -modules in  $\mathcal{C}$ . When the  $\infty$ -operad  $\mathcal{O}$  is unital and coherent, Proposition B.1.2 in [Hin15] establishes a natural equivalence from  $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$  to the underlying  $\infty$ -category of the  $\mathcal{O}$ -monoidal  $\infty$ -category  $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$  of  $A$ -modules of [Lur14, §3.3.3]. Furthermore, the following variation on the arguments of [Hin15] shows how such  $A$ -modules in the  $\infty$ -categorical sense can be strictified.

**PROPOSITION 4.4.2.** *Let  $\mathcal{M}$  be a combinatorial simplicial SM model category and let  $\mathcal{P}$  be a  $\Sigma$ -cofibrant admissible simplicial operad such that  $\mathcal{M}\mathcal{P}$  is admissible as well. For any cofibrant  $A$  in  $\mathrm{Alg}_{\mathcal{P}}(\mathcal{M})$ , there is an equivalence of  $\infty$ -categories*

$$\mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M})_{\infty} \xrightarrow{\sim} \mathrm{Mod}_A^{N(\mathcal{P})}(\mathcal{M}_{\infty}).$$

*Proof.* If  $\mathcal{P}$  is  $\Sigma$ -cofibrant and admissible, then the associated simplicial operad  $\mathcal{M}\mathcal{P}$  is  $\Sigma$ -cofibrant and admissible as well. By [PS14, Theorem 7.10], the map of operads  $\mathcal{P} \rightarrow \mathcal{M}\mathcal{P}$ , obtained as the base change of the map  $\mathrm{Com} \rightarrow \mathcal{M}\mathrm{Com}$ , induces a commuting square of  $\infty$ -categories

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})_{\infty} & \xrightarrow{\sim} & \mathrm{Alg}_{\mathcal{M}N(\mathcal{P})}(\mathcal{M}_{\infty}) \\ p \downarrow & & \downarrow q \\ \mathrm{Alg}_{\mathcal{P}}(\mathcal{M})_{\infty} & \xrightarrow{\sim} & \mathrm{Alg}_{N(\mathcal{P})}(\mathcal{M}_{\infty}) \end{array} \quad (4.4.1)$$

in which the horizontal maps are equivalences of  $\infty$ -categories. Now observe that the left vertical map  $p$  of  $\infty$ -categories is obtained by localization from the functor of relative categories

$$\pi : \mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})' \rightarrow \mathrm{Alg}_{\mathcal{P}}(\mathcal{M})^{\mathrm{cof}}$$

whose domain  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})'$  is the relative subcategory of  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})$  on those pairs  $(A, M)$  of algebras and modules whose algebra  $A$  is cofibrant. To see that the  $\infty$ -category  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})'_{\infty}$  is equivalent to  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})_{\infty}$ , note that  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})^{\mathrm{cof}}$  is a relative subcategory of  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})'$  and that the inclusion  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})^{\mathrm{cof}} \rightarrow \mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})'$  is part of a left homotopy deformation retract, with retraction given by a cofibrant replacement functor in  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})$ .

We may now identify  $\mathrm{Alg}_{\mathcal{M}\mathcal{P}}(\mathcal{M})'$  with the Grothendieck construction of the functor  $\mathrm{Mod}^{\mathcal{P}} : (\mathrm{Alg}_{\mathcal{P}}(\mathcal{M})^{\mathrm{cof}})^{\mathrm{op}} \rightarrow \mathrm{RelCat}$  sending a cofibrant  $\mathcal{P}$ -algebra  $A$  to the relative category  $\mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M})$  of  $A$ -modules and a map  $f : A \rightarrow B$  of cofibrant  $\mathcal{P}$ -algebras to the restriction functor  $f^* : \mathrm{Mod}_B^{\mathcal{P}}(\mathcal{M}) \rightarrow \mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M})$  between module categories. We note that the functor  $\mathrm{Mod}^{\mathcal{P}}$  sends weak equivalences of cofibrant algebras to equivalences of relative categories by [BM09, Theorem 2.6]. We may hence apply [Hin13, Proposition 2.1.4] to the map  $\pi^{\mathrm{op}}$  and deduce that for every cofibrant  $\mathcal{P}$ -algebra  $A$  we have an equivalence of  $\infty$ -categories

$$\mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M})_{\infty} = \pi^{-1}(A)_{\infty} \xrightarrow{\sim} p^{-1}(A) \xrightarrow{\sim} q^{-1}(A) = \mathrm{Mod}_A^{N(\mathcal{P})}(\mathcal{M}_{\infty})$$

where the second map is the induced map on fibers arising from (4.4.1), and thus an equivalence.  $\square$

**THEOREM 4.4.3.** *Let  $\mathcal{C}$  be a closed SM, differentiable presentable  $\infty$ -category and let  $\mathcal{O}^{\otimes} = N^{\otimes}(\mathcal{P})$  be the operadic nerve of a fibrant simplicial operad. Then the forgetful functor induces an equivalence of  $\infty$ -categories*

$$\mathcal{T}_A \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{T}_A \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}).$$

*Proof.* Since weakly equivalent fibrant simplicial operads have equivalent associated  $\infty$ -operads, we may assume that  $\mathcal{P}$  is  $\Sigma$ -cofibrant. By [NS15, Theorem 1.1] there exists a left proper, combinatorial simplicial SM model category  $\mathcal{M}$  together with a symmetric monoidal equivalence of  $\infty$ -categories  $(\mathcal{M}^\otimes)_\infty \simeq \mathcal{C}^\otimes$ . Furthermore,  $\mathcal{M}$  has the property that any simplicial operad is admissible [NS15, Theorem 2.5]. Since  $\mathcal{C}$  is assumed to be differentiable, the model category  $\mathcal{M}$  is differentiable as well. Consider the commutative diagram of  $\infty$ -categories

$$\begin{array}{ccccc} \mathrm{Sp}'(\mathrm{Alg}_{\mathcal{P}}(\mathcal{M})_{A//A})_\infty & \longrightarrow & \mathrm{Sp}((\mathrm{Alg}_{\mathcal{P}}(\mathcal{M})_\infty)_{A//A}) & \longrightarrow & \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{A//A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sp}'(\mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M})_{A//A})_\infty & \longrightarrow & \mathrm{Sp}((\mathrm{Mod}_A^{\mathcal{P}}(\mathcal{M})_\infty)_{A//A}) & \longrightarrow & \mathrm{Sp}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_{A//A}) \end{array}$$

where, as in the previous section, for a model category  $\mathcal{N}$  we denote by  $\mathrm{Sp}'(\mathcal{N}) \subseteq \mathcal{N}^{\mathbb{N} \times \mathbb{N}}$  the full relative subcategory spanned by the  $\Omega$ -spectra. Now the horizontal maps in the right square are equivalences, because they are equivalences before stabilization and before slice-coslicing by the rectification result of [PS14, Theorem 7.10] and by Proposition 4.4.2. The horizontal maps in the left square are equivalences by Remark 2.2.9. Finally, the left vertical map is an equivalence by Corollary 4.3.2. It then follows that the right vertical map is an equivalence, as desired.  $\square$

When  $\mathcal{C}$  is stable, the  $\infty$ -category  $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$  is stable (Remark 4.3.5) and the kernel functor of Lemma 3.4.3 yields an equivalence  $\mathcal{T}_A \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$  for every  $A$ . In this case the conclusion of Theorem 4.4.3 reduces to the following generalization of [Lur14, Theorem 7.3.4.7] to the case of  $\infty$ -operads which are not necessarily unital or coherent (but which do arise as nerves of simplicial operads):

**COROLLARY 4.4.4.** *Let  $\mathcal{C}$  be a closed SM, **stable** presentable  $\infty$ -category and let  $\mathcal{O}^\otimes = \mathcal{N}^\otimes(\mathcal{P})$  be the operadic nerve of a fibrant simplicial operad. Then the functor  $\ker : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{A//A} \rightarrow \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$  induces an equivalence of  $\infty$ -categories*

$$\mathcal{T}_A \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}).$$

## Appendix A. The filtration on a free algebra

In this appendix we will recall the natural filtration on the free algebra over a colored operad  $\mathcal{P}$  generated by an object  $X$  together with a map  $\mathcal{P}_0 \rightarrow X$ , i.e. the free  $\mathcal{P}$ -algebra where the nullary operations have already been specified. This is a special case of the filtration on a pushout of  $\mathcal{P}$ -algebras along a free map  $\mathcal{P} \circ X \rightarrow \mathcal{P} \circ Y$  (see, e.g., [PS14], [BM09] and [Cav14]) in the case where  $X = \mathcal{P}_0$  and the pushout is taken along  $\mathcal{P} \circ \mathcal{P}_0 \rightarrow \mathcal{P}_0$ . For our purposes we need a somewhat more specific formulation of these results, in which the filtration is directly associated to a natural skeletal filtration on  $\mathcal{P}$ .

In particular, while the filtration we discuss in this appendix is not new, its formulation in terms of skeletal filtration makes it fairly amenable to various manipulations, and may be of independent interest.

Let  $\mathcal{M}$  be a closed symmetric monoidal category and let  $\mathcal{P}$  be a  $W$ -colored symmetric sequence in  $\mathcal{M}$ . Recall from §4.1 that  $\mathcal{P}_{\leq n}$  is the  $W$ -colored symmetric sequence which agrees with  $\mathcal{P}$  in arities  $\leq n$  and whose higher entries are all  $\emptyset_{\mathcal{M}}$  (see Definition 4.1.5). We consider  $\mathcal{P}_{\leq n}$  as an  $n$ -**skeleton** of  $\mathcal{P}$ . Similarly, we denote by  $\mathcal{P}_n$  the symmetric sequence which agrees with  $\mathcal{P}$  in arity  $n$  and whose entries are  $\emptyset_{\mathcal{M}}$  in arities  $\neq n$ . We note that if  $\mathcal{P}$  is an operad then the 1-skeleton

$\mathcal{P}_{\leq 1}$  carries a canonical operad structure (but not the other skeleta). We will denote by  $\mathcal{O} \stackrel{\text{def}}{=} \mathcal{P}_{\leq 0}^+$  the operad freely generated from  $\mathcal{P}_{\leq 0}$ . We now recall that  $\mathcal{P}_n$  inherits from  $\mathcal{P}$  the structure of a  $\mathcal{P}_1$ -bimodule and  $\mathcal{P}_{\leq n}$  inherits from  $\mathcal{P}$  the structure of a  $\mathcal{P}_{\leq 1}$ -bimodule.

In particular, there is a canonical map  $\mathcal{P}_n \rightarrow \mathcal{P}_{\leq n}$  of left  $\mathcal{P}_1$ -modules, which induces a map  $\mathcal{P}_n \circ \mathcal{O} \rightarrow \mathcal{P}_{\leq n}$  of  $\mathcal{P}_1 - \mathcal{O}$ -bimodules.

LEMMA A.0.1. *Let  $\mathcal{P}$  be a  $W$ -colored operad in  $\mathcal{M}$ . Then for every  $n \geq 2$  there is a pushout square of  $\mathcal{P}_1 - \mathcal{O}$ -bimodules*

$$\begin{array}{ccc} (\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} & \longrightarrow & \mathcal{P}_n \circ \mathcal{O} \\ \downarrow & & \downarrow \\ \mathcal{P}_{\leq n-1} & \longrightarrow & \mathcal{P}_{\leq n}. \end{array} \quad (\text{A.0.2})$$

*Proof.* The composition product  $(X, Y) \mapsto X \circ Y$  preserves colimits in the first argument, and colimits in the second argument if  $X$  is concentrated in arity 1. This implies that the forgetful functor from  $\mathcal{P}_1 - \mathcal{O}$ -bimodules to  $W$ -symmetric sequences preserves and detects colimits, and so it suffices to show that the above square is a pushout square in the category of  $W$ -symmetric sequences. Since all objects are trivial in arities  $> n$  and both horizontal maps are isomorphisms in arities  $< n$ , it remains to prove that the square in arity  $n$  is a pushout square. Indeed, in arity  $n$  the left vertical map is an isomorphism between initial objects and the right vertical map is an isomorphism because  $\mathcal{O}$  coincides with the monoidal unit in arities  $\geq 1$ .  $\square$

Let us now consider the natural operad maps  $\mathcal{O} \xrightarrow{\psi} \mathcal{P}_{\leq 1} \xrightarrow{\varphi} \mathcal{P}$ . The inclusion  $\rho = \varphi \circ \psi : \mathcal{O} \rightarrow \mathcal{P}$  induces a free functor  $\rho_! : \text{Alg}_{\mathcal{O}} \rightarrow \text{Alg}_{\mathcal{P}}$ . When  $X$  is an  $\mathcal{O}$ -algebra (i.e. an object of  $\mathcal{M}^W$  equipped with a map from  $\mathcal{P}_0$ ),  $\rho_!(X)$  is given by the relative composition product  $\mathcal{P} \circ_{\mathcal{O}} X$  (which, as a  $W$ -colored symmetric sequence, is concentrated in arity 0). The above lemma shows that the underlying left  $\mathcal{P}_{\leq 1}$ -module of the free  $\mathcal{P}$ -algebra  $\rho_!(X)$  can be written as a colimit  $\rho_!(X) = \mathcal{P} \circ_{\mathcal{O}} X = \text{colim}_{n \geq 1} \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X$  where each step  $n \geq 2$  can be understood in terms of a pushout square of left  $\mathcal{P}_1$ -modules

$$\begin{array}{ccc} (\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_n \circ X \\ \downarrow & & \downarrow \\ \mathcal{P}_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X. \end{array} \quad (\text{A.0.3})$$

However, this filtration is somewhat non-satisfactory: while  $\mathcal{P} \circ_{\mathcal{O}} X = \text{colim}_n \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X$  is a filtration of  $\mathcal{P} \circ_{\mathcal{O}} X$  as a left  $\mathcal{P}_{\leq 1}$ -module (or a  $\mathcal{P}_{\leq 1}$ -algebra), the consecutive steps A.0.3 are only pushout squares of left  $\mathcal{P}_1$ -modules. We note that the difference between the two notions is not big. Since  $\mathcal{P}_{\leq 1} = \mathcal{P}_1^{\mathcal{P}_0}$  (see Remark 4.1.8) we see that if we consider  $\mathcal{P}_0$  as a left  $\mathcal{P}_{\leq 1}$ -module then the category of left  $\mathcal{P}_{\leq 1}$ -modules is naturally equivalent to the category

of left  $\mathcal{P}_1$ -modules under  $\mathcal{P}_0$ . We may hence fix the situation by performing a mild ‘‘cobase change’’.

DEFINITION A.0.2. Let  $X$  be an  $\mathcal{O}$ -algebra. We define the map  $R_n^-(X) \rightarrow R_n^+(X)$  by forming the following pushout diagram in the category of left  $\mathcal{P}_1$ -modules

$$\begin{array}{ccccc} (\mathcal{P}_n \circ \mathcal{O})_0 & \longrightarrow & (\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_n \circ X \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_0 & \longrightarrow & R_n^-(X) & \longrightarrow & R_n^+(X) \end{array}$$

As  $R_n^-(X)$  and  $R_n^+(X)$  are left  $\mathcal{P}_1$ -modules which carry a map of left  $\mathcal{P}_1$ -modules from  $\mathcal{P}_0$  we may naturally consider both of them as left  $\mathcal{P}_{\leq 1}$ -modules. We also remark that  $R_n^-(X)$  and  $R_n^+(X)$  are concentrated in arity 0 (since all the other objects in the square are), and we may hence consider them also as  $\mathcal{P}_{\leq 1}$ -algebras.

LEMMA A.0.3. *Let  $X$  be an  $\mathcal{O}$ -algebra. Then for every  $n \geq 2$  there is a pushout square of  $\mathcal{P}_{\leq 1}$ -algebras*

$$\begin{array}{ccc} R_n^-(X) & \longrightarrow & R_n^+(X) \\ \downarrow & & \downarrow \\ \mathcal{P}_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_{\leq n} \circ_{\mathcal{O}} X. \end{array} \quad (\text{A.0.4})$$

*Proof.* We have a commutative diagram of left  $\mathcal{P}_1$ -modules

$$\begin{array}{ccccc} (\mathcal{P}_n \circ \mathcal{O})_0 & \longrightarrow & (\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_n \circ X \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_0 & \longrightarrow & \mathcal{P}_{\leq n-1} & \longrightarrow & \mathcal{P}_{\leq n}. \end{array} \quad (\text{A.0.5})$$

Using the universal property of pushouts we may extend A.0.5 to a commutative diagram of the form

$$\begin{array}{ccccc} (\mathcal{P}_n \circ \mathcal{O})_0 & \longrightarrow & (\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ_{\mathcal{O}} X & \longrightarrow & \mathcal{P}_n \circ X \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_0 & \longrightarrow & R_n^-(X) & \longrightarrow & R_n^+(X) \\ & & \downarrow & & \downarrow \\ & & \mathcal{P}_{\leq n-1} & \longrightarrow & \mathcal{P}_{\leq n} \end{array} \quad (\text{A.0.6})$$

where the upper rectangle is the one defining  $R_n^-(X) \longrightarrow R_n^+(X)$ . The right vertical rectangle is just A.0.3, and is hence a pushout rectangle. It then follows that the bottom right square is a pushout square of left  $\mathcal{P}_1$ -modules, as desired.  $\square$

Our goal in the remainder of this section is to compute the map of symmetric sequences underlying  $R_n^-(X) \longrightarrow R_n^+(X)$  (see Corollary A.0.4). For  $w_0 \in W$  let us denote by  $\Sigma_{w_0}^n \subseteq \Sigma_W^n$  the full subgroupoid spanned by those  $\overline{w} \in \Sigma_W^n$  such that  $w_* = w_0$ . If we consider  $w_0$  as an object of  $\Sigma_W$  of arity 0 then the  $w_0$ -part of  $\mathcal{P}_n \circ X$  is simply given by

$$(\mathcal{P}_n \circ X)(w_0) = \operatorname{colim}_{\overline{w} \in \Sigma_{w_0}^n} \mathcal{P}(\overline{w}) \otimes \bigotimes_{i \in \underline{n}} X(w_i)$$

where the coproduct is taken over all isomorphism classes of  $\overline{w} \in \Sigma_{w_0}^n$ . The object  $(\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ_{\mathcal{O}} X$  has a more complicated description. By definition it is given as the coequalizer of the diagram

$$(\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ_{\mathcal{O}} X \rightrightarrows (\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ X$$

where one of the arrows is induced from the left  $\mathcal{O}$ -module structure of  $X$  and the second from the right  $\mathcal{O}$ -module structure of  $(\mathcal{P}_n \circ \mathcal{O})_{\leq n-1}$ .

Consider the full subgroupoid  $\operatorname{Dec}_{w_0}^{n-1} \subseteq \operatorname{Dec}_W$  (see §4.1) spanned by those objects  $(\phi : \underline{k} \longrightarrow \underline{n}, \overline{v} : (\underline{k} \amalg \underline{n})_+ \longrightarrow W) \in \operatorname{Dec}_W$  such that  $k \leq n-1$  and  $v_* = w_0$ . Since  $\mathcal{P}_n$  is concentrated in arity

$n$  and  $X$  is concentrated in arity 0 we may readily compute that

$$((\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ X)(w_0) = \operatorname{colim}_{\substack{(\phi, \bar{v}) \in \\ \operatorname{Dec}_{w_0}^{n-1}}} \left[ \mathcal{P}_n(\bar{v}|_{\underline{n}_+}) \otimes \bigotimes_{i \in \underline{n}} \mathcal{O}(\bar{v}|_{\phi_+^{-1}(i)}) \otimes \bigotimes_{i \in \underline{k}} X(v_i) \right]. \quad (\text{A.0.7})$$

For a fixed  $\bar{w}$  of arity  $n$ , let  $\operatorname{Dec}_{\bar{w}} \subseteq \operatorname{Dec}_{w_0}^{n-1}$  be the full subgroupoid spanned by those objects  $(\phi : \underline{k} \rightarrow \underline{n}, v)$  such that  $k \leq n-1$ ,  $\phi$  is injective and  $v$  is given by the composition  $(\underline{k} \amalg \underline{n})_+ \rightarrow \underline{n}_+ \xrightarrow{\bar{w}} W$ . Since  $\mathcal{O}$  contains only identities and 0-ary operations (the latter being the 0-ary operations of  $\mathcal{P}$ ), the colimit in A.0.7 is supported on the full subgroupoid  $\amalg_{[\bar{w}]} \operatorname{Dec}_{\bar{w}} \subseteq \operatorname{Dec}_{w_0}^{n-1}$ .

We now observe that an object  $(\phi, \bar{v})$  of  $\operatorname{Dec}_{\bar{w}}$  is completely determined, up to isomorphism, by the image  $I = \operatorname{Im}(\phi) \subsetneq \underline{n}$ , and that the automorphism group of such a  $(\phi, \bar{v})$  is exactly the subgroup of  $\operatorname{Aut}(\underline{n})$  which preserves  $I$  as a set. We may hence identify  $\operatorname{Dec}_{\bar{w}}$  with the **action groupoid** associated to the action of  $\operatorname{Aut}(\bar{w})$  on the set of **proper** subsets of  $\underline{n}$ . Our computation then unfolds as:

$$\begin{aligned} ((\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ X)(w_0) &= \operatorname{colim}_{\bar{w} \in \Sigma_{w_0}^n} \operatorname{colim}_{(\phi, \bar{v}) \in \operatorname{Dec}_{\bar{w}}} \left[ \mathcal{P}_n(\bar{w}) \otimes \bigotimes_{i \in \underline{n}} \mathcal{O}(\bar{v}|_{\phi_+^{-1}(i)}) \otimes \bigotimes_{i \in \underline{k}} X(v_i) \right] \\ &= \operatorname{colim}_{\bar{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\bar{w}) \otimes \left[ \amalg_{I \subsetneq \underline{n}} \bigotimes_{j \in \underline{n} \setminus I} \mathcal{P}_0(w_j) \otimes \bigotimes_{i \in I} X(w_i) \right] \right] \\ &= \operatorname{colim}_{\bar{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\bar{w}) \otimes \amalg_{I \subsetneq \underline{n}} \mathcal{F}_X(\bar{w}, I) \right] \end{aligned}$$

where we have set  $\mathcal{F}_X(\bar{w}, I) \stackrel{\text{def}}{=} \bigotimes_{j \in \underline{n} \setminus I} \mathcal{P}_0(w_j) \otimes \bigotimes_{i \in I} X(w_i)$ . Similarly we may compute

$$\begin{aligned} ((\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ \mathcal{O} \circ X)(w_0) &= \operatorname{colim}_{\bar{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\bar{w}) \otimes \left[ \amalg_{I \subsetneq \underline{n}} \bigotimes_{j \in \underline{n} \setminus I} \mathcal{P}_0(w_j) \otimes \bigotimes_{i \in I} (\mathcal{P}_0(w_i) \amalg X(w_i)) \right] \right] \\ &= \operatorname{colim}_{\bar{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\bar{w}) \otimes \left[ \amalg_{I' \subseteq I \subsetneq \underline{n}} \bigotimes_{j \in \underline{n} \setminus I} \mathcal{P}_0(w_j) \otimes \bigotimes_{j \in I \setminus I'} \mathcal{P}_0(w_j) \otimes \bigotimes_{i \in I'} X(w_i) \right] \right] \\ &\cong \operatorname{colim}_{\bar{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\bar{w}) \otimes \amalg_{I' \subseteq I \subsetneq \underline{n}} \mathcal{F}_X(\bar{w}, I') \right]. \end{aligned}$$

At this point it makes sense to define the object  $Q(X, \bar{w})$  to be the coequalizer of the diagram

$$\amalg_{I' \subseteq I \subsetneq \underline{n}} \mathcal{F}_X(\bar{w}, I') \rightrightarrows \amalg_{I \subsetneq \underline{n}} \mathcal{F}_X(\bar{w}, I)$$

where one of the maps sends the component  $\mathcal{F}_X(\bar{w}, I')$  to the same component on the right hand side, while the other map sends it to the component  $\mathcal{F}_X(\bar{w}, I)$  using the structure maps  $\mathcal{P}_0(w_i) \rightarrow X(w_i)$  for  $i \in I \setminus I'$ . We note that  $Q(X, \bar{w})$  carries a natural action of  $\operatorname{Aut}(\bar{w})$  and our computation above boils down to

$$((\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ X)(w_0) = \operatorname{colim}_{\bar{w} \in \Sigma_W^n(w_0)} \mathcal{P}_n(\bar{w}) \otimes Q(X, \bar{w}).$$

Finally, we note the the coequalizer defining  $Q(X, \bar{w})$  is exactly the coequalizer computing the colimit of the functor  $\operatorname{Sub}^0(\underline{n}) \rightarrow \mathcal{M}$  given by  $I \mapsto \mathcal{F}(\bar{w}, I)$ , where  $\operatorname{Sub}^0(\underline{n})$  is the poset of proper

subsets of  $\underline{n}$ . In particular, we have

$$Q(X, \overline{w}) = \operatorname{colim}_{I \in \operatorname{Sub}^0(\underline{n})} \mathcal{F}(\overline{w}, I)$$

and we may hence identify the natural map  $Q(X, \overline{w}) \rightarrow \bigotimes_{i \in \underline{n}} X(w_i)$  with the pushout-product  $f(w_i) \square \dots \square f(w_i)$  where  $f(w_i) : \mathcal{P}_0(w_i) \rightarrow X(w_i)$  is the relevant component of the structure map  $\mathcal{P}_0 \sqcup X = \mathcal{O} \circ X \rightarrow X$ .

COROLLARY A.0.4. *For each  $w_0 \in W$  we may identify the map*

$$((\mathcal{P}_n \circ \mathcal{O})_{\leq n-1} \circ \mathcal{O} \circ X)(w_0) \rightarrow (\mathcal{P}_n \circ X)(w_0)$$

*with the map*

$$\operatorname{colim}_{\overline{w} \in \Sigma_{w_0}^n} [\mathcal{P}_n(\overline{w}) \otimes Q(X, \overline{w})] \rightarrow \operatorname{colim}_{\overline{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\overline{w}) \otimes \bigotimes_{i \in \underline{n}} X(w_i) \right] \quad (\text{A.0.8})$$

*induced by the pushout-product  $f(w_i) \square \dots \square f(w_i) : Q(X, \overline{w}) \rightarrow \bigotimes_{i \in \underline{n}} X(w_i)$ . For each  $w_0 \in W$  we may hence identify the map  $\varphi_{w_0} : R_n^-(X)(w_0) \rightarrow R_n^+(X)(w_0)$  with the cobase change of A.0.8 along the map*

$$\operatorname{colim}_{\overline{w} \in \Sigma_{w_0}^n} \left[ \mathcal{P}_n(\overline{w}) \otimes \bigotimes_{i \in \underline{n}} \mathcal{P}_0(w_i) \right] \rightarrow \mathcal{P}_0(w_0)$$

## REFERENCES

- ABG11 M. Ando, A. J. Blumberg, D. Gepner, *Parametrized spectra, multiplicative Thom spectra, and the twisted Umkehr map*, preprint arXiv:1112.2203, 2011.
- Bar07 C. Barwick, *On Reedy model categories*, preprint arXiv:0708.2832, 2007.
- Bec67 J. Beck, *Triples, algebras and cohomology*, Ph.D. thesis, Columbia University, 1967, Reprints in Theory and Applications of Categories, 2, 2003, p. 1–59.
- BHH16 I. Barnea, Y. Harpaz, G. Horel, *Pro-categories in homotopy theory*, Algebraic and Geometric Topology, to appear.
- BM05 M. Basterra and M. A. Mandell, *Homology and cohomology of  $E_\infty$  ring spectra*, Mathematische Zeitschrift 249(4), pp. 903–944 (2005).
- BM09 C. Berger, I. Moerdijk. *On the derived category of an algebra over an operad*, Georgian Mathematical Journal, 16(1), p. 13–28 (2009).
- Cav14 G. Caviglia, *A model structure for enriched coloured operads*, preprint arXiv:1401.6983, 2014.
- CHH16 H. Chu, R. Haugseng, G. Heuts, *Two models for the homotopy theory of  $\infty$ -operads*, preprint arXiv:1606.03826, 2016.
- Dug01 D. Dugger, *Combinatorial model categories have presentations*, Advances in Mathematics, 1.164, 2001, p. 177–201.
- Fre09 B. Fresse, *Modules over operads and functors*, Springer, 2009.
- Hel97 A. Heller, *Stable homotopy theories and stabilization*, Journal of Pure and Applied Algebra, 115.2, 1997, p. 113–130.
- HHM15 G. Heuts, V. Hinich, I. Moerdijk, *On the equivalence between Lurie’s model and the dendroidal model for infinity-operads*, preprint arXiv:1305.3658, 2013.
- Hin13 V. Hinich, *Dwyer-Kan localization revisited*, preprint arXiv:1311.4128 (2013).
- Hin15 V. Hinich, *Rectification of algebras and modules*, Documenta Mathematica 20, 2015, p. 879–926.
- Hir03 P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, xvi+457 pp. (2003).
- HNP16 Y. Harpaz, J. Nuiten, M. Prasma, *The abstract cotangent complex and Quillen cohomology of enriched categories*, preprint, 2016.
- Hov99 M. Hovey, *Model categories*, No. 63. American Mathematical Soc., 1999.
- Hov01 M. Hovey, *Spectra and symmetric spectra in general model categories*, Journal of Pure and Applied Algebra, 165.1, 2001, p. 63–127.
- HP15 Y. Harpaz and M. Prasma, *The Grothendieck construction for model categories*, Advances in Mathematics, 2015.
- Lur06 J. Lurie, *Stable infinity categories*, arXiv preprint math/0608228 (2006).
- Lur09 J. Lurie, *Higher topos theory*, No. 170. Princeton University Press, (2009).
- Lur14 J. Lurie, *Higher Algebra*, preprint, available at Author’s Homepage (2011).
- Lyd98 M. Lydakis, *Simplicial functors and stable homotopy theory*, Sonderforschungsbereich 343, 1998.
- MS06 J. P. May, J. Sigurdsson, *Parametrized homotopy theory*. No. 132. American Mathematical Soc., 2006.
- NS15 T. Nikolaus, S. Sagave, *Presentably symmetric monoidal infinity-categories are represented by symmetric monoidal model categories*, preprint arXiv:1506.01475, 2015.
- PS14 D. Pavlov, J. Scholbach, *Admissibility and rectification of colored symmetric operads*, preprint arXiv:1410.5675, 2014.
- Qui70 D. Quillen, *On the (co-)homology of commutative rings*, Proc. Symp. Pure Math. Vol. 17. No. 2. 1970.
- Shi04 B. Shipley, *A convenient model category for commutative ring spectra*, Contemporary Mathematics, 346, 2004, p. 473–484.

- Rez02 C. Rezk, *Every homotopy theory of simplicial algebras admits a proper model*, Topology and its Applications, 119.1, 2002, p. 65–94.
- Rob12 M. Robalo, *Noncommutative motives i: A universal characterization of the motivic stable homotopy theory of schemes*, preprint arXiv:1206.3645 (2012).
- RR15 G. Raptis, J. Rosický, *The accessibility rank of weak equivalences*, Theory and Applications of Categories, 30.19, 201, p. 687–703.
- Sch97 S. Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, Journal of Pure and Applied Algebra, 120.1, 1997, p. 77–104.

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